Particle supported control of a fluid-particle system

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Abstract

In this paper we study, from a control theoretic view point, a 1D model of fluid-particle interaction. More precisely, we consider a point mass moving in a pipe filled with a fluid. The fluid is modelled by the viscous Burgers equation whereas the point mass obeys Newton's second law. The control variable is a force acting on the mass point. The main result of the paper asserts that for any initial data there exist a time $T > 0$ and a control such that, at the end of the control process, the particle reaches a point arbitrarily close to a given target, whereas the velocities of the fluid and of the point mass are driven exactly to zero. Therefore, within this simplified model, we can control simultaneously the fluid and the particle, by using inputs acting on the moving point only. Moreover, the main result holds without any smallness assumptions on the initial data. Alternatively, we can see our results as yielding controllability of the viscous Burgers equation by a moving internal boundary.

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1 Introduction

The study of the equations modeling the motion of rigid bodies in a viscous incompressible fluid became an active research area in the last two decades. Early references (see, for instance, [8, 10, 22, 23]) where devoted to existence theory of the corresponding initial value problem.

As far as we know, the problem of control and stabilization of such complex systems coupling the interactions between a fluid and a structure by inputs acting only on the immersed body is at an embrionary stage. Some stabilization results, requiring smoothness and smallness of the initial data, have been given in Takahashi et al. [25]. Controllability of this models is a clearly challenging mathematical question since positive results would imply that, in some sense, the water in a pool can be controlled by forces acting only on the immersed body. From the applications viewpoint the obtained control strategy could provide a methodology for stealthy motion of bodies immersed in a fluid or techniques for the control of waves makers.

In this paper we consider a 1-d model for fluid-solid interaction which has been introduced by Vasquez and Zuazua in [28, 29]. In these articles the authors studied the global existence of solutions and their large time behavior. Later on, the boundary controllability problem for this system was addressed by Doubova and Fernandez-Cara [11]. The authors showed the null-controllability of the coupled system by using controls acting on both extremities of the domain of the fluid. The methodology used in [11], combining global Carleman estimates and fixed point techniques, has been extended to the two-dimensional case in Imanuvilov and Takahashi [16] and, independently, in Boulakia and Osses [3]. The main question left open in the one dimensional case studied in [11] consisted in establishing the null controllability when the control acts at one end only. A positive answer to this question has been given in Liu, Takahashi and Tucsnak [19] by combining spectral methods and a new fixed point procedure.

In this paper we consider the simplified model already studied in [11] and [19] but the control problem we study is a different one. More precisely, the main novelty is that the control is active only on the moving particle. We have thus to tackle, besides the typical features of nonlinear coupled problems, the difficulties specific to pointwise control problems for PDE's, the most important one coming from the presence of nodal points. One of the ways to overcome the effects of nodal points consists in using moving actuators as, for instance, in Khapalov [17] or Castro and Zuazua [6] (see also Demetriou and Hussein [9], Rosier and Zhang [20], Chavez-Silva, Rosier and Zuazua [7] for problems involving distributed moving actuators). The system we consider shares with those in [6, 17] the fact that the control is supported in a moving point but differs of these systems by the presence of a free boundary and by the fact that our aim consists not only in controlling the solution of the PDE but also the position of the actuator.

More precisely, we consider the following system, which can be seen as a model for the motion of a particle, under the action of an exterior force, in a one-dimensional fluid:

$$\begin{cases} \dot{v}(t,y) - v_{yy}(t,y) + v(t,y)v_y(t,y) = 0 & t \in (0,T),\ y \in (-1,1),\ y \neq h(t), \\ v(t,-1) = v(t,1) = 0 & t \in (0,T), \\ \dot{h}(t) = v(t,h(t)) & t \in (0,T), \\ \ddot{h}(t) = [v_y](t,h(t)) + u(t) & t \in (0,T), \\ v(0,y) = v_0(y) & y \in (-1,1), \\ h(0) = h_0, \qquad \dot{h}(0) = g_0. \end{cases} \tag{1.1}$$

In (1.1), $v = v(t,y)$ denotes the eulerian velocity field of the fluid filling the interval $(-1,1)$ whereas $h = h(t)$ indicates the position of the point mass and the derivative with respect to time is denoted by a dot. We assume that the velocity v of the fluid is governed by the viscous Burgers equation on both sides of the moving mass. The fourth equation in (1.1) is the second Newton's law applied to the mass. The forces acting on the point mass are due to the fluid (the jump of the derivative of v when crossing the mass which is denoted by $[v_y](t,h(t))$) and to the exterior input $u(t)$. For the sake of simplicity, we have assumed that the mass of the body, the viscosity and the density of the fluid are equal to one.

Note that (1.1) is a free boundary value problem since $h(t)$, delimiting the intervals in which the Burgers equation holds, is one of the unknowns of the problem. The presence of a free boundary requires an appropriate definition of the notion of finite energy solution, which reads as follows:

Definition 1.1. *Given $T>0$, $v_0 \in L^2[-1,1]$, $h_0 \in (-1,1)$, $g_0 \in \mathbb{R}$ and $u \in L^2[0,T]$, we say that*

$$\begin{bmatrix} v \\ g \\ h \end{bmatrix} \in \left\{\mathcal{C}([0,T]; L^2[-1,1]) \cap L^2([0,T]; \mathcal{H}_0^1(-1,1))\right\} \times L^2[0,T] \times \mathcal{H}^1(0,T),$$

is a finite energy solution of (1.1) on $[0,T]$ if $h(0)=h_0$, $\dot{h}(t) = g(t) = v(t,h(t))$ and $h(t) \in (-1,1)$, for almost every $t \in [0,T]$ and

$$\begin{aligned} \int_{-1}^{1} v(t,y)\psi(t,y)\,\mathrm{d}y - \int_{-1}^{1} v_0(y)\psi(0,y)\,\mathrm{d}y + g(t)l(t) - g_0 l(0) - \int_0^t g(\sigma)\dot{l}(\sigma)\,\mathrm{d}\sigma \\ - \int_0^t \int_{-1}^{1} v(\sigma,y)\dot{\psi}(\sigma,y)\,\mathrm{d}y\,\mathrm{d}\sigma + \int_0^t \int_{-1}^{1} v_y(\sigma,y)\psi_y(\sigma,y)\,\mathrm{d}y\,\mathrm{d}\sigma \\ - \frac{1}{2}\int_0^t \int_{-1}^{1} v^2(\sigma,y)\psi_y(\sigma,y)\,\mathrm{d}y\,\mathrm{d}\sigma = \int_0^t u(\sigma)l(\sigma)\,\mathrm{d}\sigma, \end{aligned} \tag{1.2}$$

for every $t \in [0,T]$ and for every

$$\begin{bmatrix} \psi \\ l \end{bmatrix} \in \left\{\mathcal{H}^1((0,T); L^2[-1,1]) \cap L^2([0,T]; \mathcal{H}_0^1(-1,1))\right\} \times \mathcal{H}^1(0,T), \tag{1.3}$$

$$l(t) = \psi(t,h(t)) \qquad (t \in [0,T]). \tag{1.4}$$

Note that the test functions used above depend on the solution and more precisely on its component h. In Definition 1.1 and in the rest of the paper, for each $m \geqslant 1$, $\mathcal{H}^m$ and $\mathcal{H}_0^m$ denote the classical Sobolev spaces and $\mathcal{H}^{-m}$ denotes the topological dual of $\mathcal{H}_0^m$.

The main result of this paper asserts that the mass point can be driven arbitrarily close to a given destination, whereas the velocities of the fluid and of the particle simultaneously vanish. More precisely, we have the following result.

Theorem 1.2. *Let $v_0 \in L^2[-1,1]$, $g_0 \in \mathbb{R}$ and $h_0 \in (-1,1)$. Then for every $h_F \in (-1,1)$ and $\eta > 0$ there exist $T>0$ and $u \in L^\infty[0,T]$ such that the solution (in the sense of Definition 1.1) of (1.1) satisfies*

$$v(T,\cdot) = 0, \qquad |h(T) - h_F| \leqslant \eta, \qquad \dot{h}(T) = 0. \tag{1.5}$$

Independently from the fluid-particle system, the above theorem can be interpreted as a null controllability result for the Burgers equation with a scanning actuator. This result could be stated as follows: for every $v_0 \in L^2[-1,1]$ there exists a control time $T>0$ and a control $h \in \mathcal{H}^1(0,T)$ such that the solution v of the three first equations in (1.1) with $v(0,y) = v_0(y)$ satisfies $v(T,\cdot) = 0$.

The strategy used to prove Theorem 1.2 consists of a preliminary choice and three main steps. The preliminary choice is to select an irrational algebraic number h_1 such that $|h_F - h_1| < \eta$. The first main step is to give u in a feedback form for which an appropriate Lyapounov functions is non increasing along the trajectories of the obtained closed loop system. This feedback, which will be described in details in Section 4, is given by a force which is what would be produced by a spring and a damper connecting the point mass to h_1.

The second main step is to show that the proposed feedback law steers the system, when t goes to infinity, arbitrarily close to the state $\begin{bmatrix} 0 \\ 0 \\ h_1 \end{bmatrix}$. This is done by using an appropriate Lyapunov function, compactness of trajectories and Barbalat-type results.

The last main step, technically the most involved one, consists in proving local exact controllability to the equilibrium state $\begin{bmatrix} 0 \\ 0 \\ h_1 \end{bmatrix}$. To show this controllability result we perform a change of variables to a fixed spatial domain and we linearize the system around the target.

At this stage it clearly appears the necessity of choosing an exact target h_1 in the dense set $\mathcal{S}$ of algebraic irrational numbers (see Remark 7.2). Indeed, in every nonempty open interval, there exist targets which are not reachable in finite time for this linearized system. However, if the target h_1 is chosen in $\mathcal{S}$, we are able to prove that we can drive exactly the position of the body to h_1. Consequently, the controllability cost depends in a highly *instable manner* on the choice of the targets, becoming infinite if the target is rational. With the choice made for h_1 the controllability problem is tackled by transforming it into an equivalent moment problem and by constructing an explicit solution to the latter through biorthogonal techniques. To pass from the linear problem to the nonlinear one we use a fixed point method and a technique introduced in [19] to control parabolic systems with appropriate non-homogeneous terms. We think that the instability related to the choice of h_1 is due to our linearization technique used in the proof and it is not intrinsic to the original problem.

The methods used for proving the controllability result are clearly of one dimensional nature. A similar result for the corresponding three dimensional model (a Navier-Stokes fluid with a rigid body immersed in it) seems very unlikely. Indeed, this would imply that we can steer solutions of the Navier-Stokes system exactly to zero by using a finite dimensional input space.

Note that the controllability time T in Theorem 1.2 could be very large, depending on the initial data to be controlled. Obtaining a control time which is uniform for all initial data in the energy space seems unlikely. This is suggested by the fact that null-controllability in uniform time is not valid for the viscous Burger's equation with a boundary control (see, for instance, [13, Theorem 6.4, p. 61] or [12]).

Several technical results are gathered in the appendices. Appendix A is devoted to the study of the spectral properties of the generator of a linearized system and Appendix B contains a construction of a biorthogonal family which may be of larger interest in the study of systems coupling PDE's and ODE's.

2 Change of variables and transformed equations

In this section we introduce a change of variables which allows us to write system (1.1) in an equivalent form, but with the involved PDE written in a fixed spatial domain. The standard way to accomplish this goal consists in introducing, for each $p \in (-1,1)$ a strictly increasing homeomorphism $\Psi(\cdot,p)$ of $[-1,1]$ such that $\Psi(p,p) = h_0$. In the case of system (1.1), this idea has already been used in [11] and [19], where $\Psi(\cdot,p)$ has been simply chosen to be affine on $[-1,p]$ and on $[p,1]$. However, since our notion of solution is weaker than the one in [11] and [19], it seems that we need smoother transformations Ψ and with $\Psi_y(y,p) = 1$ for y in a neighborhood of p. Therefore, we adapt below the more involved construction used in Takahashi [24] in the analysis of the system modeling the motion of rigid bodies in a Navier-Stokes flow.

Given $\varepsilon > 0$ and p, $h_0 \in [-1+2\varepsilon, 1-2\varepsilon]$, we define the map

$$\Lambda(y,p) = (p - h_0)\vartheta(y) \qquad y \in [-1,1],$$

where $\vartheta \in \mathcal{D}(-1,1)$ verifies $\vartheta(y) = 1$, for $y \in [-1+\varepsilon, 1-\varepsilon]$ and $\vartheta(y) = 0$, for $y \notin \left[-1+\frac{\varepsilon}{2}, 1-\frac{\varepsilon}{2}\right]$. Consider the initial value problem

$$\begin{cases} \widetilde{\Psi}_s(s;y,p) = \Lambda(\widetilde{\Psi}(s;y,p),p), \qquad s \in [0,1] \\ \widetilde{\Psi}(0;y,p) = y. \end{cases} \tag{2.1}$$

Since the map $y \mapsto (p-h_0)\vartheta(y)$ is C^∞, the initial value problem (2.1) admits, for every $y \in [-1,1]$ a unique solution $\widetilde{\Psi}(s;y,p)$ with $s \in [0,\varrho]$, for some $\varrho > 0$. Since ϑ vanishes outside the interval $[-1+\frac{\varepsilon}{2}, 1-\frac{\varepsilon}{2}]$, we obtain that $\widetilde{\Psi}(s;y,p) \in [-1,1]$ for every $s \in [0,\varrho]$, $y \in [-1,1]$ and $p \in [-1+2\varepsilon, 1-2\varepsilon]$. This implies, in particular, that the $\widetilde{\Psi}$ can be extended to a solution of (2.1) defined for every $s \geqslant 0$. We define $\Psi(y,p) = \widetilde{\Psi}(1,y,p)$. The main properties of map Ψ are summarized in the result below.

Lemma 2.1. *We have $\Psi \in C^\infty([-1,1]\times(-1+\varepsilon,1-\varepsilon))$ and, for every $p \in [-1+2\varepsilon, 1-2\varepsilon]$, the map $y \mapsto \Psi(y,p)$ is a diffeomorphism from $[-1,1]$ onto itself, from $[-1,p]$ onto $[-1,h_0]$ and from $[p,1]$ onto $[h_0,1]$. Moreover, we have that*

$$\Psi(y,p) = y - p + h_0 \qquad (y \in [-1+\varepsilon, 1-\varepsilon]) \tag{2.2}$$

$$\Psi(y,p) = y \qquad \left(y \notin \left[-1+\frac{\varepsilon}{2}, 1-\frac{\varepsilon}{2}\right]\right). \tag{2.3}$$

For every $p \in [-1+\varepsilon, 1-\varepsilon]$, the inverse map $x \mapsto \Phi(x,p)$ of $y \mapsto \Psi(y,p)$ is in $C^\infty([-1,1]\times(-1+\varepsilon,1-\varepsilon))$. Finally, we have

$$\Phi_x(x,p) \geqslant e^{-|p-h_0|K_1(\varepsilon)} \qquad (x \in [-1,1],\ p \in [-1+\varepsilon, 1-\varepsilon]), \tag{2.4}$$

$$\Phi_x(x,h_0) = 1, \quad \Phi_{xx}(x,h_0) = 0 \qquad (x \in [-1,1]), \tag{2.5}$$

where $K_1(\varepsilon) = \|\vartheta_x\|_{\mathcal{C}[-1,1]}$.

Proof. The fact that $\Psi \in C^\infty([-1,1]\times(-1+\varepsilon,1-\varepsilon))$ is a consequence of classical results for ODE's (see, for instance, Hartman [15, p. 100]).

To prove (2.2), let $y \in [p-\varepsilon, p+\varepsilon]$. Since $p, h_0 \in [-1+2\varepsilon, 1-2\varepsilon]$, we have that $y - s(p-h_0) \in [-1+\varepsilon, 1-\varepsilon]$ for every $s \in [0,1]$. Consequently, the function $s \mapsto y - s(p-h_0)$ is the solution of initial value problem (2.1) for any $y \in [p-\varepsilon, p+\varepsilon]$, i.e. we have (2.2). Similar estimates lead to (2.3).

We note that the function $x \mapsto \Phi(x,p)$ with $\Phi(x,p) = \widetilde{\Phi}(0,x,p)$, where $\widetilde{\Phi}$ is the solution of the final value (backwards) problem

$$\begin{cases} \widetilde{\Phi}_s(s,x,p) = (p-h_0)\vartheta(\widetilde{\Phi}(s,x,p)) \\ \widetilde{\Phi}(1,x,p) = x, \end{cases} \tag{2.6}$$

is, by Cauchy-Lipschitz Theorem, the inverse of the map $y \mapsto \Psi(y,p)$. To prove (2.4), we note that, according to the well-known results (see, for instance, [15, p. 95]), the function $s \mapsto \widetilde{\Phi}_x(s,x,p)$ verifies

$$\begin{cases} \widetilde{\Phi}_{sx}(s,x,p) = (p-h_0)\vartheta_x(\widetilde{\Phi}(s,x,p))\widetilde{\Phi}_x(s,x,p) \\ \widetilde{\Phi}_x(1,x,p) = 1. \end{cases} \tag{2.7}$$

Looking to the above equation as an initial value problem of unknown $s \mapsto \widetilde{\Phi}_x(s,x,p)$, we deduce (2.4). Finally, (2.5) are direct consequences of (2.2). □

Given $\varepsilon > 0$ and a function $h \in \mathcal{H}^1(0,T)$, such that $h(t) \in [-1+2\varepsilon, 1-2\varepsilon]$ for all $t \in [0,T]$, we introduce the change of variables $w(t,x) = v(t,y)$, where

$$\begin{cases} y = \Phi(x,h(t)), \\ x = \Psi(y,h(t)). \end{cases} \tag{2.8}$$

Remark 2.2. *The properties of the functions Ψ and Φ from Lemma 2.1 and the fact that $h \in \mathcal{H}^1(0,T)$ implies that the application*

$$\mathcal{T} : \mathcal{H}^1((0,T); L^2[-1,1]) \cap L^2([0,T]; \mathcal{H}_0^1(-1,1)) \to \mathcal{H}^1((0,T); L^2[-1,1]) \cap L^2([0,T]; \mathcal{H}_0^1(-1,1)),$$

$$\mathcal{T}(\psi)(t,x) = \psi(t, \Phi(x, h(t))),$$

is a well defined one to one map whose inverse is given by

$$\mathcal{T}^{-1}(\varphi)(t,y) = \varphi(t, \Psi(y, h(t))).$$

The following proposition uses the change of variable (2.8) to rewrite system (1.1) in a fixed spatial domain.

Proposition 2.3. *Let $T > 0$, $v_0 \in L^2[-1,1]$, $h_0 \in (-1,1)$, $g_0 \in \mathbb{R}$, $u \in L^2[0,T]$ and a triplet of functions*

$$\begin{bmatrix} v \\ g \\ h \end{bmatrix} \in \left\{ \mathcal{C}([0,T]; L^2[-1,1]) \cap L^2([0,T]; \mathcal{H}_0^1(-1,1)) \right\} \times L^2[0,T] \times \mathcal{H}^1(0,T). \tag{2.9}$$

Then $\begin{bmatrix} v \\ g \\ h \end{bmatrix}$ is a finite energy solution of (1.1) on $[0,T]$ if and only if, the triplet $\begin{bmatrix} w \\ g \\ h \end{bmatrix}$, where $w(t,x) = v\left(t, \Phi(x, h(t))\right)$ for every $t \in [0,T]$, $h(0) = h_0$, $\dot{h}(t) = g(t) = w(t, h_0)$ and $h(t) \in [-1,1]$, for almost every $t \in [0,T]$ verifies the relation

$$\begin{aligned}
\int_{-1}^{1} w(t,x)\varphi(t,x)\Phi_x(x,h(t))\,\mathrm{d}x - \int_{-1}^{1} v_0(x)\varphi(0,x)\,\mathrm{d}x + g(t)l(t) - g_0 l(0) \\
- \int_0^t \int_{-1}^{1} w(\sigma,x)\dot{\varphi}(\sigma,x)\Phi_x(x,h(\sigma))\,\mathrm{d}x\,\mathrm{d}\sigma + \int_0^t g(\sigma) \int_{-1}^{1} w(\sigma,x)\varphi_x(\sigma,x)\Phi_p(x,h(\sigma))\,\mathrm{d}x\,\mathrm{d}\sigma \\
- \int_0^t g(\sigma)\dot{l}(\sigma)\,\mathrm{d}\sigma + \int_0^t \int_{-1}^{1} \frac{1}{\Phi_x(x,h(\sigma))} w_x(\sigma,x)\varphi_x(\sigma,x)\,\mathrm{d}x\,\mathrm{d}\sigma \\
- \frac{1}{2}\int_0^t \int_{-1}^{1} w^2(\sigma,x)\varphi_x(\sigma,x)\,\mathrm{d}x\,\mathrm{d}\sigma = \int_0^t u(\sigma)l(\sigma)\,\mathrm{d}\sigma,
\end{aligned} \tag{2.10}$$

for every $t \in [0,T]$ and for every

$$\begin{bmatrix} \varphi \\ l \end{bmatrix} \in \left\{ \mathcal{H}^1((0,T); L^2[-1,1]) \cap L^2([0,T]; \mathcal{H}_0^1(-1,1)) \right\} \times \mathcal{H}^1(0,T), \tag{2.11}$$

$$l(t) = \varphi(t, h_0) \qquad (t \in [0,T]). \tag{2.12}$$

Proof. Let $\begin{bmatrix} v \\ g \\ h \end{bmatrix}$ be a triplet of functions satisfying (2.9) and assume that $\begin{bmatrix} \varphi \\ l \end{bmatrix}$ satisfies (2.11) and (2.12). From Remark 2.2 it follows that $\begin{bmatrix} \psi \\ l \end{bmatrix}$ given by $\psi(t,y) = \varphi(t, \Psi(y, h(t)))$ verifies (1.3) and (1.4).

Using the change of variables $y = \Phi(x, h(t))$ and noting that $\varphi(t,x) = \psi\left(t, \Phi(x, h(t))\right)$ in the first two integrals appearing in (1.2) we obtain

$$\int_{-1}^{1} v(t,y)\psi(t,y)\,\mathrm{d}y = \int_{-1}^{1} w(t,x)\varphi(t,x)\Phi_x(x,h(t))\,\mathrm{d}x \qquad (t \in [0,T]), \tag{2.13}$$

$$\int_{-1}^{1} v_0(y)\psi(0,y)\,\mathrm{d}y = \int_{-1}^{1} v_0(x)\varphi(0,x)\,\mathrm{d}x \qquad (t\in[0,T]). \tag{2.14}$$

On the other hand, since $\psi(t,y) = \varphi\left(t,\Psi(y,h(t)\right)$ it follows that

$$\dot\psi(t,y) = \dot\varphi(t,\Psi(y,h(t)) + \varphi_x\left(t,\Psi(y,h(t)\right)\Psi_p(y,h(t))\dot h(t) \qquad (t\in[0,T],\quad y\in[-1,1]).$$

Consequently, setting again $y = \Phi(x,h(t))$ and $\varphi(t,x) = \psi\left(t,\Phi(x,h(t)\right)$, we get

$$\int_{-1}^{1} v(\sigma,y)\dot\psi(\sigma,y)\,\mathrm{d}y = \int_{-1}^{1} w(\sigma,x)\dot\varphi(\sigma,x)\Phi_x(x,h(\sigma))\,\mathrm{d}x \\ - \dot h(\sigma)\int_{-1}^{1} w(\sigma,x)\varphi_x(\sigma,x)\Phi_p(x,h(\sigma))\,\mathrm{d}x \qquad (\sigma\in[0,T]). \tag{2.15}$$

Similar calculations show that

$$\int_{-1}^{1} v_y(\sigma,y)\psi_y(\sigma,y)\,\mathrm{d}y = \int_{-1}^{1} \frac{1}{\Phi_x(x,h(\sigma))} w_x(\sigma,x)\varphi_x(\sigma,x)\,\mathrm{d}x \qquad (\sigma\in[0,T]), \tag{2.16}$$

$$\int_{-1}^{1} v^2(\sigma,y)\psi_y(\sigma,y)\,\mathrm{d}y = \int_{-1}^{1} w^2(\sigma,x)\varphi_x(\sigma,x)\,\mathrm{d}x. \tag{2.17}$$

Putting together (2.13)-(2.17) we obtain that if $\begin{bmatrix} v\\ g\\ h\end{bmatrix}$ is a finite energy solution of (1.1) then $\begin{bmatrix} w\\ g\\ h\end{bmatrix}$ verifies (2.10) for every $\begin{bmatrix} \varphi\\ l\end{bmatrix}$ satisfying (2.11)-(2.12) and $\dot h(t) = g(t) = w(t,h_0)$ for almost every $t\in[0,T]$.

From Remark 2.2 and by using similar arguments we deduce that the converse assertion holds too. $\square$

The above proposition implies, using the fact that $\Phi_x(x,h_0) = 1$ for every $x\in[-1,1]$, the following result.

Corollary 2.4. *Let $T>0$, $v_0\in L^2[-1,1]$, $g_0\in\mathbb{R}$, $h_0\in(-1,1)$, $u\in L^2[0,T]$ and a triplet of functions*

$$\begin{bmatrix} v\\ g\\ h\end{bmatrix} \in \left\{\mathcal{C}([0,T];L^2[-1,1])\cap L^2([0,T];\mathcal{H}_0^1(-1,1))\right\}\times L^2[0,T]\times\mathcal{H}^1(0,T). \tag{2.18}$$

Then $\begin{bmatrix} v\\ g\\ h\end{bmatrix}$ is a finite energy solution of (1.1) on $[0,T]$ if and only if, the triplet $\begin{bmatrix} z\\ g\\ h\end{bmatrix}$, where the function z is given by $z(t,x) = \Phi_x(x,h(t))v\left(t,\Phi(x,h(t))\right)$ for every $t\in[0,T]$, and the function h satisfies $h(0)=h_0$, $\dot h(t) = g(t) = z(t,h_0)$ and $h(t)\in[-1,1]$, for almost every $t\in[0,T]$, verifies the relation

$$\int_{-1}^{1} z(t,x)\varphi(t,x)\,\mathrm{d}x - \int_{-1}^{1} z_0(x)\varphi(0,x)\,\mathrm{d}x + g(t)l(t) - g_0 l(0) \\ - \int_0^t\int_{-1}^{1} z(\sigma,x)\dot\varphi(\sigma,x)\,\mathrm{d}x\,\mathrm{d}\sigma + \int_0^t g(\sigma)\int_{-1}^{1} z(\sigma,x)\varphi_x(\sigma,x)\frac{\Phi_p(x,h(\sigma))}{\Phi_x(x,h(\sigma))}\,\mathrm{d}x\,\mathrm{d}\sigma \\ - \int_0^t g(\sigma)\dot l(\sigma)\,\mathrm{d}\sigma + \int_0^t\int_{-1}^{1} \frac{z_x(\sigma,x)}{(\Phi_x(x,h(\sigma)))^2}\varphi_x(\sigma,x)\,\mathrm{d}x\,\mathrm{d}\sigma - \int_0^t\int_{-1}^{1} \frac{z(\sigma,x)\Phi_{xx}(x,h(\sigma))}{(\Phi_x(x,h(\sigma)))^3}\varphi_x(\sigma,x)\,\mathrm{d}x\,\mathrm{d}\sigma \\ - \frac{1}{2}\int_0^t\int_{-1}^{1} \frac{z^2(\sigma,x)}{(\Phi_x(x,h(\sigma)))^2}\varphi_x(\sigma,x)\,\mathrm{d}x\,\mathrm{d}\sigma = \int_0^t u(\sigma)l(\sigma)\,\mathrm{d}\sigma, \tag{2.19}$$

for every $t\in[0,T]$ and for every $\begin{bmatrix} \varphi\\ l\end{bmatrix}$ satisfying (2.11)-(2.12).

3 Study of a linear operator

An important role in the remaining part of this paper is played by a self-adjoint operator which we introduce below. Consider the Hilbert space

$$H = L^2[-1,1] \times \mathbb{R},$$

endowed with the inner product

$$\left\langle \begin{bmatrix} \varphi_1 \\ p_1 \end{bmatrix}, \begin{bmatrix} \varphi_2 \\ p_2 \end{bmatrix} \right\rangle = \int_{-1}^{1} \varphi_1(x)\varphi_2(x)\,\mathrm{d}x + p_1 p_2. \tag{3.1}$$

The norm in H will be denoted by $\| \ \|$. We define the unbounded operator $A_0 : \mathcal{D}(A_0) \to H$,

$$\mathcal{D}(A_0) = \left\{ \begin{bmatrix} \varphi \\ p \end{bmatrix} \in \mathcal{H}_0^1(-1,1) \times \mathbb{R} \;\middle|\; \varphi_{|(-1,h_0)} \in \mathcal{H}^2(-1,h_0), \varphi_{|(h_0,1)} \in \mathcal{H}^2(h_0,1),\ \varphi(h_0) = p \right\}, \tag{3.2}$$

$$A_0 \begin{bmatrix} \varphi \\ p \end{bmatrix} = \begin{bmatrix} -\{\varphi_{xx}\}_{h_0} \\ -[\varphi_x]_{h_0} \end{bmatrix} \qquad \left(\begin{bmatrix} \varphi \\ p \end{bmatrix} \in \mathcal{D}(A_0) \right). \tag{3.3}$$

In the last formula $\{\varphi_{xx}\}_{h_0}$ stands for the function in $L^2[-1,1]$ defined, for almost every $x \in [-1,1]$, by

$$\{\varphi_{xx}\}_{h_0}(x) = \begin{cases} \psi_1(x) & x \in [-1,h_0] \\ \psi_2(x) & x \in [h_0,1] \end{cases}$$

where ψ_1 (respectively ψ_2) is the second derivative of φ in $\mathcal{D}'(-1,h_0)$ (respectively in $\mathcal{D}'(h_0,1)$). This function is connected to the derivative in the sense of $\mathcal{D}'(-1,1)$, denoted as usually by φ_{xx}, and to the jump of φ_x at h_0, denoted $[\varphi_x]_{h_0}$, via the jump formula

$$\varphi_{xx} = \{\varphi_{xx}\}_{h_0} + [\varphi_x]_{h_0}\delta_{h_0} \qquad (\varphi \in \mathcal{D}(A_0)). \tag{3.4}$$

Proposition 3.1. *The operator A_0 is self-adjoint and strictly positive in H. The operator $-A_0$ is the generator of a contraction semigroup $\mathbb{T}$ in H. Moreover, the corresponding space $H_{\frac{1}{2}}$ (i.e., $\mathcal{D}(A_0^{\frac{1}{2}})$ endowed with the graph norm of $A_0^{\frac{1}{2}}$) is*

$$H_{\frac{1}{2}} = \left\{ \begin{bmatrix} \varphi \\ p \end{bmatrix} \in \mathcal{H}_0^1(-1,1) \times \mathbb{R} \,\middle|\, \varphi(h_0) = p \right\}, \tag{3.5}$$

endowed with the product

$$\left\langle \begin{bmatrix} \varphi_1 \\ p_1 \end{bmatrix}, \begin{bmatrix} \varphi_2 \\ p_2 \end{bmatrix} \right\rangle_{\frac{1}{2}} = \int_{-1}^{1} \varphi_{1,x}(x)\varphi_{2,x}(x)\,\mathrm{d}x. \tag{3.6}$$

The dual space $H_{-\frac{1}{2}}$ of $H_{\frac{1}{2}}$ with respect to the pivot space H, is given by $H_{-\frac{1}{2}} = W$, where W is the quotient space of $\mathcal{H}^{-1}(-1,1) \times \mathbb{R}$ with respect to its closed subspace spanned by $\left\{ \begin{bmatrix} \delta_{h_0} \\ -1 \end{bmatrix} \right\}$. Denoting by $\widehat{\begin{bmatrix} \psi \\ \alpha \end{bmatrix}}$ the equivalence class of $\begin{bmatrix} \psi \\ \alpha \end{bmatrix} \in \mathcal{H}^{-1}(-1,1) \times \mathbb{R}$, this quotient space is endowed with the norm

$$\left\| \widehat{\begin{bmatrix} \psi \\ \alpha \end{bmatrix}} \right\|_{-\frac{1}{2}} = \|\psi + \alpha\delta_{h_0}\|_{\mathcal{H}^{-1}(-1,1)}, \tag{3.7}$$

and the corresponding duality product writes, for all $\widehat{\begin{bmatrix} \psi \\ \alpha \end{bmatrix}} \in H_{-\frac{1}{2}}$, $\begin{bmatrix} \varphi \\ l \end{bmatrix} \in H_{\frac{1}{2}}$,

$$\left\langle \widehat{\begin{bmatrix} \psi \\ \alpha \end{bmatrix}}, \begin{bmatrix} \varphi \\ l \end{bmatrix} \right\rangle_{-\frac{1}{2},\frac{1}{2}} = \langle \psi, \varphi \rangle_{\mathcal{H}^{-1}(-1,1),\mathcal{H}_0^1(-1,1)} + \alpha l. \tag{3.8}$$

Proof. We first check that A_0 is symmetric. Indeed, for any $\Phi_i = \begin{bmatrix} \varphi_i \\ p_i \end{bmatrix} \in \mathcal{D}(A_0)$, $i = 1, 2$, we have that

$$\langle A_0\Phi_1, \Phi_2\rangle = -\int_{-1}^{h_0} \varphi_{1,xx}(x)\varphi_2(x)\,\mathrm{d}x - \int_{h_0}^{1} \varphi_{1,xx}(x)\varphi_2(x)\,\mathrm{d}x - [\varphi_{1,x}]_{h_0}p_2$$
$$= \int_{-1}^{1} \varphi_{1,x}(x)\varphi_{2,x}(x)\,\mathrm{d}x = \langle \Phi_1, A_0\Phi_2\rangle. \quad (3.9)$$

Taking $\Phi_1 = \Phi_2 = \Phi = \begin{bmatrix} \varphi \\ p \end{bmatrix}$ in (3.9) we see that,

$$\langle A_0\Phi, \Phi\rangle = \int_{-1}^{1} \varphi_x^2(x)\,\mathrm{d}x \qquad \left(\begin{bmatrix} \varphi \\ p \end{bmatrix} \in \mathcal{D}(A_0)\right), \quad (3.10)$$

so that A_0 is a monotone operator.

We next check that A_0 is onto. For $F = \begin{bmatrix} f \\ g \end{bmatrix} \in H$, the equation $A_0\Phi = F$, of unknown $\Phi = \begin{bmatrix} \varphi \\ p \end{bmatrix} \in \mathcal{D}(A_0)$, writes

$$\begin{cases} -\{\varphi_{xx}\}_{h_0}(x) = f(x) & x \in (-1, h_0) \cup (h_0, 1) \\ \varphi(h_0) = p & \\ -[\varphi_x]_{h_0} = g. & \end{cases} \quad (3.11)$$

Let $-\widetilde{A_0}$ be the Dirichlet Laplacian on $(-1, 1)$, i.e. the operator

$$\widetilde{A_0}\varphi = -\varphi_{xx} \qquad (\varphi \in \mathcal{H}_0^1(-1, 1)),$$

which is a continuous isomorphism from $\mathcal{H}_0^1(-1, 1)$ onto $\mathcal{H}^{-1}(-1, 1)$. Using (3.4), we see that (3.11) writes

$$\widetilde{A_0}\varphi = f + g\delta_{h_0}.$$

Consequently, for every $\begin{bmatrix} f \\ g \end{bmatrix} \in H$, there exists a unique solution $\begin{bmatrix} \varphi \\ p \end{bmatrix} \in H_1$ of (3.11) given by

$$\varphi = \widetilde{A_0}^{-1} f + gG, \qquad p = \left(\widetilde{A_0}^{-1} f\right)(h_0) + g\frac{1 - h_0^2}{1 + h_0^2}, \quad (3.12)$$

where

$$G(x) = \begin{cases} \frac{(1-h_0)(1+x)}{2} & \text{for } \ x \in [-1, h_0], \\ \frac{(1+h_0)(1-x)}{2} & \text{for } x \in [h_0, 1]. \end{cases}$$

We have shown that indeed A_0 is onto. Since we have already shown that A_0 is symmetric, classical results (see, for instance, [27, Proposition 3.2.4 and Theorem 3.8.4]) implies that A_0 is self-adjoint and $-A_0$ is the generator of a contraction semigroup $\mathbb{T}$ in H.

On the other hand, (3.10), Poincaré's inequality and a simple trace theorem imply that that the square root of the right-hand side of (3.10) defines on $\mathcal{D}(A_0)$ a norm which is equivalent with the standard norm on $\mathcal{H}_0^1(-1, 1) \times \mathbb{R}$. Consequently A_0 is indeed strictly positive and $H_{\frac{1}{2}}$ is given by (3.5) with the inner product defined by (3.6).

To prove the facts asserted on $H_{-\frac{1}{2}}$, we first note that $H_{\frac{1}{2}}$ is a closed subspace of

$$W = \mathcal{H}_0^1(-1, 1) \times \mathbb{R},$$

whose dual space (with respect to the pivot space H) is obviously $\mathcal{H}^{-1}(-1,1)\times\mathbb{R}$. It is not difficult to check that the annihilator of W (using again the pivot space H) is

$$\left(H_{\frac{1}{2}}\right)^{\perp} = \operatorname{Span}\left\{\begin{bmatrix}\delta_{h_0}\\ -1\end{bmatrix}\right\} \subset \mathcal{H}^{-1}(-1,1)\times\mathbb{R}.$$

Consequently, according to to a classical result (see, for instance, [21, Theorem 4.9]), the dual space of $H_{\frac{1}{2}}$ with respect to the pivot space H is given by the quotient space

$$H_{-\frac{1}{2}} = \left(\mathcal{H}^{-1}(-1,1)\times\mathbb{R}\right)\Big/\left(H_{\frac{1}{2}}\right)^{\perp}.$$

To prove (3.7), note that

$$\left\|\widehat{\begin{bmatrix}\psi\\ \alpha\end{bmatrix}}\right\|_{-\frac{1}{2}} = \inf_{\beta\in\mathbb{R}}\left\|\begin{bmatrix}\psi\\ \alpha\end{bmatrix} - \beta\begin{bmatrix}\delta_{h_0}\\ -1\end{bmatrix}\right\|_{\mathcal{H}^{-1}(-1,1)\times\mathbb{R}} =$$

$$= \inf_{\beta\in\mathbb{R}}\left\{\|\psi-\beta\delta_{h_0}\|_{\mathcal{H}^{-1}(-1,1)} + |\alpha+\beta|\right\} \leqslant \|\psi+\alpha\delta_{h_0}\|_{\mathcal{H}^{-1}(-1,1)}.$$

On the other hand, by taking into account that $\|\delta_{h_0}\|_{\mathcal{H}^{-1}(-1,1)}\leqslant 1$, we have that

$$\begin{aligned}\inf_{\beta\in\mathbb{R}}&\left\{\|\psi-\beta\delta_{h_0}\|_{\mathcal{H}^{-1}(-1,1)} + |\alpha+\beta|\right\}\\ &\geqslant \inf_{\beta\in\mathbb{R}}\left\{\|\psi+\alpha\delta_{h_0}\|_{\mathcal{H}^{-1}(-1,1)} - \|(\alpha+\beta)\delta_{h_0}\|_{\mathcal{H}^{-1}(-1,1)} + |\alpha+\beta|\right\}\\ &\geqslant \inf_{\beta\in\mathbb{R}}\|\psi+\alpha\delta_{h_0}\|_{\mathcal{H}^{-1}(-1,1)} = \|\psi+\alpha\delta_{h_0}\|_{\mathcal{H}^{-1}(-1,1)}.\end{aligned}$$

Hence, (3.7) holds.

□

Remark 3.2. For the sake of simplicity we denote, for the remaining part of this work, the duality between $H_{-\frac{1}{2}}$ and $H_{\frac{1}{2}}$ by $\left\langle\begin{bmatrix}\psi\\ \alpha\end{bmatrix},\begin{bmatrix}\varphi\\ l\end{bmatrix}\right\rangle_{-\frac{1}{2},\frac{1}{2}}$ instead of $\left\langle\widehat{\begin{bmatrix}\psi\\ \alpha\end{bmatrix}},\begin{bmatrix}\varphi\\ l\end{bmatrix}\right\rangle_{-\frac{1}{2},\frac{1}{2}}$ from (3.8).

The main result of this section is the following

Proposition 3.3. *With the notation in Proposition 3.1, let $T>0$. Then, for any $\begin{bmatrix}z_0\\ g_0\end{bmatrix}\in H$ and $\begin{bmatrix}f_1\\ f_2\end{bmatrix}\in L^2([0,T],H_{-\frac{1}{2}})$, there exists a unique function*

$$\begin{bmatrix}z\\ g\end{bmatrix}\in\mathcal{C}([0,T];H)\cap L^2([0,T];H_{\frac{1}{2}})\cap\mathcal{H}^1((0,T),H_{-\frac{1}{2}}), \tag{3.13}$$

such that $z(0)=z^0$, $g(0)=g^0$ and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1} z(t,x)\varphi(t,x)\,\mathrm{d}x - \langle\dot\varphi(t,\cdot),z(t,\cdot)\rangle_{\mathcal{H}^{-1}(-1,1),\mathcal{H}_0^1(-1,1)} + \frac{\mathrm{d}}{\mathrm{d}t}(g(t)l(t)) - g(t)\dot l(t)\\ + \int_{-1}^{1} z_x(t,x)\varphi_x(t,x)\,\mathrm{d}x = f_2(t)l(t) + \langle f_1(t),\varphi(t,\,\cdot\,)\rangle_{\mathcal{H}^{-1}(-1,1),\mathcal{H}_0^1(-1,1)},\end{aligned} \tag{3.14}$$

for every $\begin{bmatrix}\varphi\\ l\end{bmatrix} \in L^2([0,T];H_{\frac{1}{2}}) \cap \mathcal{H}^1((0,T);H_{-\frac{1}{2}})$ *and for almost every* $t \in [0,T]$*. Moreover, for any* $t \in [0,T]$*, we have that*

$$\|z(t,\cdot)\|^2_{L^2[-1,1]} + |g(t)|^2 + \int_0^t \|z_x(\sigma,\,\cdot)\|^2_{L^2[-1,1]}\,\mathrm{d}\sigma$$

$$\leqslant \|z_0\|^2_{L^2[-1,1]} + |g_0|^2 + \left\|\begin{bmatrix}f_1\\ f_2\end{bmatrix}\right\|^2_{L^2([0,T];H_{-\frac{1}{2}})}. \tag{3.15}$$

Proof. The existence and uniqueness of $\begin{bmatrix}z\\ g\end{bmatrix}$ satisfying (3.13) and (3.14) and the energy estimate (3.15) are consequences of classical results for parabolic problems (see, for instance, Zeidler [31, Propositions 23.3 and 23.23]), applied to the specific operator A_0 considered in Proposition 3.1. $\square$

4 Local wellposedness of a closed loop problem

In this section we consider equations (1.1) with

$$u(t) = -k_v\dot{h}(t) + k_p(h_1 - h(t)) \qquad (t \in [0,T]), \tag{4.1}$$

where $k_v \geqslant 0$ and $k_p > 0$ are fixed constants and $h_1 \in (-1,1)$. With this feedback law, the total energy of the system is non increasing. This will be proved rigourously in the next section but we briefly justify this choice by formal calculations below. Indeed, assume that v and h are smooth functions satisfying (1.1). Multiplying the terms in the first equation by v and integrating on $(-1,h(t))$ and on $(h(t),1)$, it is easily checked that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1} v^2\mathrm{d}y = -\int_{-1}^{1} v_y^2\mathrm{d}y - [v_y](t,h(t))v(t,h(t)).$$

On the other hand, multiplying the fourth equation in (1.1) with $\dot{h}$ it follows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\dot{h}(t))^2 = [v_y](t,h(t))\dot{h}(t) + u(t)\dot{h}(t).$$

Summing up the last two formulas it follows that, for u given by (4.1), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1} v^2\,\mathrm{d}y + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\dot{h}(t))^2 + \frac{k_p}{2}\frac{\mathrm{d}}{\mathrm{d}t}|h(t)-h_1|^2 = -\int_{-1}^{1} v_y^2\,\mathrm{d}y - k_v\dot{h}^2(t), \tag{4.2}$$

so that the energy of the system is indeed non increasing.

The main result of this section states as follows.

Theorem 4.1. *For* $\kappa > 0$ *and* $\varepsilon > 0$ *we denote by* $\mathcal{B}_{\kappa,\varepsilon}$ *the set of* $\begin{bmatrix}v_0\\ g_0\\ h_0\end{bmatrix} \in H \times [-1,1]$ *satisfying*

$$\|v_0\|^2_{L^2[-1,1]} + |g_0|^2 < \kappa^2, \tag{4.3}$$

$$|h_0| \leqslant 1 - 4\varepsilon. \tag{4.4}$$

Then there exists $T > 0$*, depending only on* κ *and* ε*, such that for every* $\begin{bmatrix}v_0\\ g_0\\ h_0\end{bmatrix} \in \mathcal{B}_{\kappa,\varepsilon}$ *system* (1.1)*, with* u *given by* (4.1)*, admits a unique solution* $\begin{bmatrix}v\\ g\\ h\end{bmatrix}$*, in the sense of Definition* 1.1*, on the time* *interval* $[0,T]$. *Moreover, the map*

$$\begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix} \mapsto \begin{bmatrix} v \\ g \\ h \end{bmatrix}, \tag{4.5}$$

is continuous from $\mathcal{B}_{\kappa,\varepsilon}$ *to* $\left\{\mathcal{C}([0,T];L^2[-1,1]) \cap L^2([0,T];\mathcal{H}_0^1(-1,1))\right\} \times L^2[0,T] \times \mathcal{H}^1(0,T)$.

An important ingredient for the proof of Theorem 4.1 are the properties of the operators $\mathcal{G}_k$, with $k \in \{1,2,3,4\}$, which are defined (as suggested by (2.19)) by

$$\left\langle \mathcal{G}_1 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}(t), \begin{bmatrix} \varphi \\ l \end{bmatrix} \right\rangle_{-\frac{1}{2},\frac{1}{2}} = \int_{-1}^{1} z_x(t,x) \left(1 - \frac{1}{(\Phi_x(x,h(t)))^2}\right) \varphi_x(x)\,\mathrm{d}x, \tag{4.6}$$

$$\left\langle \mathcal{G}_2 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}(t), \begin{bmatrix} \varphi \\ l \end{bmatrix} \right\rangle_{-\frac{1}{2},\frac{1}{2}} = \int_{-1}^{1} z(t,x) \left(\frac{\Phi_{xx}(x,h(t))}{(\Phi_x(x,h(t)))^3} - \frac{\Phi_p(x,h(t))}{\Phi_x(x,h(t))} g(t)\right) \varphi_x(x)\,\mathrm{d}x, \tag{4.7}$$

$$\left\langle \mathcal{G}_3 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}(t), \begin{bmatrix} \varphi \\ l \end{bmatrix} \right\rangle_{-\frac{1}{2},\frac{1}{2}} = \frac{1}{2}\int_{-1}^{1} z^2(t,x) \frac{1}{\Phi_x^2(x,h(t))} \varphi_x(x)\,\mathrm{d}x, \tag{4.8}$$

$$\left\langle \mathcal{G}_4 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}(t), \begin{bmatrix} \varphi \\ l \end{bmatrix} \right\rangle_{-\frac{1}{2},\frac{1}{2}} = (-k_v g(t) + k_p(h_1 - h(t)))l, \tag{4.9}$$

for every $\begin{bmatrix} \varphi \\ l \end{bmatrix} \in H_{\frac{1}{2}}$, where z, g satisfy (3.14) and $h(t) = h_0 + \int_0^t g(s)\,\mathrm{d}s$. Note that the operators $\mathcal{G}_k$ depend on v_0, g_0 and h_0 but, in order to simplify the notation, we omit for the moment this dependence.

Also, we remark that $\begin{bmatrix} z \\ g \end{bmatrix} \in \mathcal{C}([0,T];H) \cap L^2([0,T];H_{\frac{1}{2}})$ verifies (2.19) if it is a solution of (3.14) with a second member given by

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = (\mathcal{G}_1 + \mathcal{G}_2 + \mathcal{G}_3 + \mathcal{G}_4) \begin{bmatrix} f_1 \\ f_2 \end{bmatrix},$$

or, equivalently $\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ is a fixed point of $\mathcal{G}_1 + \mathcal{G}_2 + \mathcal{G}_3 + \mathcal{G}_4$.

We give below some of the properties of the operators $(\mathcal{G}_k)_{1 \leqslant k \leqslant 4}$.

Lemma 4.2. *For any* $T > 0$ *the operators* $(\mathcal{G}_k)_{1 \leqslant k \leqslant 4}$ *given by* (4.6)-(4.9) *are well defined maps from* $L^2([0,T];H_{-\frac{1}{2}})$ *to itself. Moreover, assume that for some* κ, $\varepsilon > 0$ *we have*

$$\left\| \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right\|_{L^2([0,T];H_{-\frac{1}{2}})} \leqslant \kappa, \qquad \left\| \begin{bmatrix} z_0 \\ g_0 \end{bmatrix} \right\| \leqslant \kappa, \quad 1 - |h_0| \geqslant 4\varepsilon. \tag{4.10}$$

Then there exists a constant $K(\varepsilon) > 0$ *such that for every* $T \leqslant \min\left\{\frac{\sqrt{2}\varepsilon}{\kappa}, 1\right\}$ *we have*

$$1 - |h(t)| \geqslant 2\varepsilon \qquad (t \in [0,T]), \tag{4.11}$$

$$\left\| \mathcal{G}_k \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right\|_{L^2([0,T];H_{-\frac{1}{2}})} \leqslant K(\varepsilon) \sqrt[4]{T} \kappa^2 \qquad (k \in \{1,2,3\}), \tag{4.12}$$

$$\left\| \mathcal{G}_4 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right\|_{L^2([0,T];H_{-\frac{1}{2}})} \leqslant K(\varepsilon) \sqrt{T} \left(|h_1 - h_0| + \kappa\right). \tag{4.13}$$

Proof. Within this proof and in the remaining part of this section we denote by $K(\varepsilon)$ a generic positive constant depending only on ε.

In order to prove (4.11) we note that, using (3.15) and taking $T \leqslant \frac{\sqrt{2}\varepsilon}{\kappa}$, we have

$$|h(t)| \leqslant |h_0| + \int_0^T |g(\sigma)|\,\mathrm{d}\sigma \leqslant 1 - 4\varepsilon + T\sqrt{2}\kappa \leqslant 1 - 2\varepsilon \qquad (t \in [0,T]).$$

From (2.4), (2.5) and the fact that $\Phi \in C^\infty([-1,1] \times [-1+2\varepsilon, 1-2\varepsilon])$, we have

$$\frac{1}{\Phi_x(x,h(t))} \leqslant e^{|h(t)-h_0|K_1(\varepsilon)} \leqslant K(\varepsilon) \qquad (t \in [0,T]), \tag{4.14}$$

$$\left|\frac{\Phi_p(x,h(t))}{\Phi_x(x,h(t))}\right| \leqslant K(\varepsilon), \tag{4.15}$$

$$\left|1 - \frac{1}{(\Phi_x(x,h(t)))^2}\right| = \left|\frac{(\Phi_x(x,h(t)))^2 - (\Phi_x(x,h_0))^2}{(\Phi_x(x,h(t)))^2}\right| \leqslant K(\varepsilon)|h(t) - h_0|, \tag{4.16}$$

$$\left|\frac{\Phi_{xx}(x,h(t))}{(\Phi_x(x,h(t)))^3}\right| = \left|\frac{\Phi_{xx}(x,h(t)) - \Phi_{xx}(x,h_0)}{(\Phi_x(x,h(t)))^3}\right| \leqslant K(\varepsilon)|h(t) - h_0|. \tag{4.17}$$

To prove (4.12) for $k = 1$, we use (4.14), (4.16) and (3.15) and we obtain the following estimates

$$\begin{aligned}
\int_0^T \left\|\mathcal{G}_1 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}(t)\right\|_{-\frac{1}{2}}^2 \mathrm{d}t &\leqslant \int_0^T \int_{-1}^1 \left|z_x(t,x)\left(1 - \frac{1}{(\Phi_x(x,h(t)))^2}\right)\right|^2 \mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T \int_{-1}^1 |z_x(t,x)|^2 \left|\frac{(\Phi_x(x,h(t)))^2 - (\Phi_x(x,h_0))^2}{(\Phi_x(x,h(t)))^2}\right|^2 \mathrm{d}x\,\mathrm{d}t \\
&\leqslant K(\varepsilon) \int_0^T (h(t) - h_0)^2 \int_{-1}^1 |z_x(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\leqslant K(\varepsilon) \int_0^T t \int_0^t |g(s)|^2\,ds \int_{-1}^1 |z_x(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant K(\varepsilon)T^2\kappa^4.
\end{aligned}$$

In order to prove (4.12) for $k = 2$ we use (4.14), (4.15), (4.17) and again (3.15). We deduce that, for $T \leqslant \min\left\{\frac{\sqrt{2}\varepsilon}{\kappa}, 1\right\}$, the following estimates hold

$$\begin{aligned}
\int_0^T \left\|\mathcal{G}_2 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}(t)\right\|_{-\frac{1}{2}}^2 \mathrm{d}t &\leqslant \int_0^T \int_{-1}^1 \left|z(t,x)\left(\frac{\Phi_{xx}(x,h(t))}{(\Phi_x(x,h(t)))^3} - \frac{\Phi_p(x,h(t))}{\Phi_x(x,h(t))}g(t)\right)\right|^2 \mathrm{d}x\,\mathrm{d}t \\
&\leqslant K(\varepsilon) \int_0^T \int_{-1}^1 |z(t,x)|^2 \left(|g(t)|^2 + |h(t) - h_0|^2\right) \mathrm{d}x\,\mathrm{d}t \leqslant K(\varepsilon)T\kappa^4.
\end{aligned}$$

To prove (4.12) for $k = 3$ we note that by using (4.14) we have

$$\int_0^T \left\|\mathcal{G}_3 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}(t)\right\|_{-\frac{1}{2}}^2 \mathrm{d}t \leqslant K(\varepsilon) \int_0^T \int_{-1}^1 z^4(t,x)\,\mathrm{d}x\,\mathrm{d}t.$$

The above inequality, the continuous embedding $\mathcal{H}^{\frac{1}{4}}(-1,1) \subset L^4(-1,1)$ (see, for instance,[1, Theorem 7.58] and an interpolation inequality (resulting from the relation $\mathcal{H}^{\frac{1}{4}} = \left[L^2, \mathcal{H}^1\right]_{\frac{1}{4}}$, see [18, Chapter 1, Section 9]) it follows that

$$\int_0^T \left\|\mathcal{G}_3 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}(t)\right\|_{-\frac{1}{2}}^2 \mathrm{d}t \leqslant K(\varepsilon) \int_0^T \|z(t,\cdot)\|_{L^2[-1,1]}^3 \|z(t,\cdot)\|_{\mathcal{H}_0^1(-1,1)}\,\mathrm{d}t.$$

Using (3.15) and the Cauchy Schwarz inequality, it follows that indeed we have (4.12) for $k = 3$.

Finally, let us prove (4.13). From (3.15), we deduce that, for $T \leqslant \min\left\{\frac{\sqrt{2}\varepsilon}{\kappa}, 1\right\}$, we have

$$\begin{aligned}\int_0^T \left\|\mathcal{G}_4 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} (t)\right\|_{-\frac{1}{2}}^2 \mathrm{d}t &\leqslant K(\varepsilon) \int_0^T |-k_v g(t) + k_p(h_1 - h(t))|^2 \,\mathrm{d}t \\ &\leqslant K(\varepsilon) \int_0^T \left(|g(t)|^2 + |h_1 - h_0|^2 + \left(\int_0^t g(s)\,\mathrm{d}s\right)^2\right) \mathrm{d}t \leqslant K(\varepsilon) T \left(\kappa^2 + |h_1 - h_0|^2\right).\end{aligned}$$

This concludes the proof of the lemma. $\square$

Lemma 4.3. *With the notation and assumptions in Lemma 4.2, we suppose that* $\begin{bmatrix} \widetilde{f}_1 \\ \widetilde{f}_2 \end{bmatrix}$ *belongs to* $L^2([0,T]; H_{-\frac{1}{2}})$ *and verifies*

$$\left\|\begin{bmatrix} \widetilde{f}_1 \\ \widetilde{f}_2 \end{bmatrix}\right\|_{L^2([0,T];H_{-\frac{1}{2}})} \leqslant \kappa. \tag{4.18}$$

Then there exists a constant $K(\varepsilon) > 0$ *such that for every* $T \leqslant \min\{\frac{\sqrt{2}\varepsilon}{\kappa}, 1\}$ *we have that*

$$\left\|\mathcal{G}_k \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} - \mathcal{G}_k \begin{bmatrix} \widetilde{f}_1 \\ \widetilde{f}_2 \end{bmatrix}\right\|_{L^2([0,T];H_{-\frac{1}{2}})} \leqslant K(\varepsilon)\, \sqrt[4]{T}\kappa(\kappa+1) \left\|\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} - \begin{bmatrix} \widetilde{f}_1 \\ \widetilde{f}_2 \end{bmatrix}\right\|_{L^2([0,T];H_{-\frac{1}{2}})} \quad (k \in \{1,2,3\}), \tag{4.19}$$

$$\left\|\mathcal{G}_4 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} - \mathcal{G}_4 \begin{bmatrix} \widetilde{f}_1 \\ \widetilde{f}_2 \end{bmatrix}\right\|_{L^2([0,T];H_{-\frac{1}{2}})} \leqslant K(\varepsilon)\, \sqrt{T} \left\|\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} - \begin{bmatrix} \widetilde{f}_1 \\ \widetilde{f}_2 \end{bmatrix}\right\|_{L^2([0,T];H_{-\frac{1}{2}})}. \tag{4.20}$$

The proof of the above Lemma is based on estimates which are very close to those used in proving Lemma 4.2, so that we omit the details.

We are now in a position to prove the main result in this section.

Proof of Theorem 4.1. For $T > 0$, κ, $\varepsilon > 0$ we set

$$\mathcal{X}_{T,\kappa} = \left\{\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in L^2([0,T]; H_{-\frac{1}{2}}) \;\middle|\; \left\|\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}\right\|_{L^2([0,T];H_{-\frac{1}{2}})} \leqslant \kappa\right\}.$$

Let $\mathcal{N} : L^2([0,T]; H_{-\frac{1}{2}}) \times \mathcal{B}_{\kappa,\varepsilon} \to L^2([0,T]; H_{-\frac{1}{2}})$ be defined by

$$\mathcal{N}\left(\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix}\right) = \mathcal{G}_1 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \mathcal{G}_2 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \mathcal{G}_3 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \mathcal{G}_4 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix},$$

where $(\mathcal{G}_k)_{1\leqslant k\leqslant 4}$ have been defined in (4.6)-(4.9).

From Lemma 4.2 it follows that, for any $T \leqslant \min\left\{1, \frac{\sqrt{2}\varepsilon}{\kappa}\right\}$, we have that

$$\left\|\mathcal{N}\left(\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix}\right)\right\|_{L^2([0,T];H_{-\frac{1}{2}})} \leqslant K(\varepsilon) \sqrt[4]{T}(\kappa^2 + 1) \qquad \left(\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in \mathcal{X}_{T,\kappa}, \quad \begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix} \in \mathcal{B}_{\kappa,\varepsilon}\right).$$

The last estimate implies that, for every $\begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix} \in \mathcal{B}_{\kappa,\varepsilon}$, the function

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \mapsto \mathcal{N}\left(\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix} \right) \tag{4.21}$$

invariates $\mathcal{X}_{T,\kappa}$, provided that

$$T \leqslant \min\left\{1, \frac{\sqrt{2}\varepsilon}{\kappa}, \left[\frac{K(\varepsilon)(\kappa^2+1)}{\kappa}\right]^{-4}\right\}. \tag{4.22}$$

By applying Lemma 4.3 it follows that, for every $T \leqslant \min\left\{1, \frac{\sqrt{2}\varepsilon}{\kappa}\right\}$ and $\begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix} \in \mathcal{B}_{\kappa,\varepsilon}$, we have

$$\begin{aligned} &\left\| \mathcal{N}\left(\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix} \right) - \mathcal{N}\left(\begin{bmatrix} \widetilde{f}_1 \\ \widetilde{f}_2 \end{bmatrix}, \begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix} \right) \right\|_{L^2([0,T];H_{-\frac{1}{2}})} \\ &\qquad \leqslant K(\varepsilon)\sqrt[4]{T}\left(1+\kappa^2\right) \left\| \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} - \begin{bmatrix} \widetilde{f}_1 \\ \widetilde{f}_2 \end{bmatrix} \right\|_{L^2([0,T];H_{-\frac{1}{2}})} \qquad \left(\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} \widetilde{f}_1 \\ \widetilde{f}_2 \end{bmatrix} \in \mathcal{X}_{T,\kappa} \right). \end{aligned}$$

The last estimate implies that the application defined in (4.21) is, for every $\begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix} \in \mathcal{B}_{\kappa,\varepsilon}$, a strict contraction of $\mathcal{X}_{T,\kappa}$, provided that

$$T \leqslant \min\left\{1, \frac{\sqrt{2}\varepsilon}{\kappa}, \frac{1}{16}\left[K(\varepsilon)(1+\kappa^2)\right]^{-4}\right\}. \tag{4.23}$$

Consequently, for every T satisfying (4.22) and (4.23) we have that $\mathcal{N}$ has a unique fixed point $\begin{bmatrix} \widehat{f}_1 \\ \widehat{f}_2 \end{bmatrix}$. Moreover, since the contraction constant of $\mathcal{N}$ depends only on ε and κ, it follows (see, for instance, Brooks and Schmitt [4, Theorem 3.8]) that the map

$$\begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix} \mapsto \begin{bmatrix} \widehat{f}_1 \\ \widehat{f}_2 \end{bmatrix} \tag{4.24}$$

is continuous from $\mathcal{B}_{\kappa,\varepsilon}$ to $L^2([0,T];H_{-\frac{1}{2}})$. Denoting by $\begin{bmatrix} z \\ g \end{bmatrix} \in \mathcal{C}([0,T];H) \cap L^2([0,T];H_{\frac{1}{2}}) \cap \mathcal{H}^1((0,T),H_{-\frac{1}{2}})$ the corresponding solution of (3.14) and taking $h(t) = h_0 + \int_0^t g(s)\,\mathrm{d}s$, it follows that

$$\begin{bmatrix} z \\ g \\ h \end{bmatrix} \in \left\{\mathcal{C}([0,T];H) \cap L^2([0,T];H_{\frac{1}{2}}) \cap \mathcal{H}^1((0,T),H_{-\frac{1}{2}})\right\} \times \mathcal{H}^1(0,T)$$

is the unique solution of (2.19). Moreover, the continuity of the function defined in (4.24) and (3.15) imply that the map

$$\begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix} \mapsto \begin{bmatrix} z \\ g \\ h \end{bmatrix},$$

is continuous from $\mathcal{B}_{\kappa,\varepsilon}$ to $\left\{\mathcal{C}([0,T];L^2[-1,1])\cap L^2([0,T];\mathcal{H}_0^1(-1,1))\right\}\times L^2[0,T]\times\mathcal{H}^1(0,T)$. According to Corollary 2.4, we obtain that $\begin{bmatrix} v\\ g\\ h\end{bmatrix}$, with $v(t,y)=\Psi_y(y,h(t))z(t,\Psi(y,h(t)))$ satisfies (1.2), so it is the unique finite energy solution of (1.1) on $[0,T]$.

Moreover, the continuity of the function defined in (4.24) implies that the map from (4.5) is continuous from $\mathcal{B}_{\kappa,\varepsilon}$ to $\left\{\mathcal{C}([0,T];L^2[-1,1])\cap L^2([0,T];\mathcal{H}_0^1(-1,1))\right\}\times L^2[0,T]\times\mathcal{H}^1(0,T)$. This ends the proof of Theorem 4.1. □

5 Global solutions of a closed loop problem

In this section we continue to study equations (1.1) with the feedback law (4.1). More precisely, we show that, under appropriate assumptions on h_0, h_1 and on the constant k_p, the local solutions constructed in the previous section can be extended to global ones. The main result of this section reads as follows.

Theorem 5.1. *Let $v_0\in L^2[-1,1]$, $g_0\in\mathbb{R}$ and $h_0\in(-1,1)$. Moreover, assume that the constants h_1 and k_p in* (4.1) *verify*

$$0<|h_1-h_0|<\frac{1}{2\sqrt{2}}\min\left(1-h_1,1+h_1\right),\tag{5.1}$$

$$k_p\geqslant\frac{\|v_0\|^2_{L^2[-1,1]}+|g_0|^2}{|h_0-h_1|^2}.\tag{5.2}$$

Then equations (1.1) *with u given by* (4.1) *admit, for every $T>0$, a unique finite energy solution $\begin{bmatrix} v\\ g\\ h\end{bmatrix}$ on $[0,T]$, such that*

$$\min\left(1-h(t),1+h(t)\right)\geqslant\frac{1}{2}\min\left(1-h_1,1+h_1\right)\qquad(t\in[0,T]).\tag{5.3}$$

Moreover, the map defined in (4.5) *(which makes now sense sense for every $T>0$) is continuous from $L^2[-1,1]\times\mathbb{R}\times(-1,1)$ to $\left\{\mathcal{C}([0,T];L^2[-1,1])\cap L^2([0,T];\mathcal{H}_0^1(-1,1))\right\}\times L^2[0,T]\times\mathcal{H}^1(0,T)$.*

To prove the above theorem we need an auxiliary result, which asserts that the energy identity (4.2), derived by formal calculations in Section 4, can be justified in a rigorous manner.

Proposition 5.2. *Let $\begin{bmatrix} v\\ g\\ h\end{bmatrix}$ be the solution of* (1.1) *constructed in Theorem 4.1. Then, for almost every $t\in[0,T]$, we have*

$$\begin{aligned}\frac{1}{2}\int_{-1}^1 v^2(t,y)\mathrm{d}y+\frac{1}{2}g^2(t)+\frac{k_p}{2}(h(t)-h_1)^2\\ =\frac{1}{2}\int_{-1}^1 v_0^2(y)\mathrm{d}y+\frac{1}{2}g_0^2+\frac{k_p}{2}(h_0-h_1)^2-\int_0^t\int_{-1}^1 v_y^2(\sigma,y)\mathrm{d}y-k_v\int_0^t g^2(\sigma)\,\mathrm{d}\sigma.\end{aligned}\tag{5.4}$$

Proof. Since $\begin{bmatrix} v\\ g\end{bmatrix}\in\mathcal{C}([0,T];H)\cap L^2([0,T];H_{\frac{1}{2}})\cap\mathcal{H}^1((0,T),H_{-\frac{1}{2}})$, from (1.2) and a density argument we deduce that

$$\left\langle \begin{bmatrix} v \\ g \end{bmatrix}(t), \begin{bmatrix} \psi \\ l \end{bmatrix}(t) \right\rangle - \left\langle \begin{bmatrix} v_0 \\ g_0 \end{bmatrix}, \begin{bmatrix} \psi \\ l \end{bmatrix}(0) \right\rangle$$
$$- \int_0^t \left\langle \begin{bmatrix} v \\ g \end{bmatrix}(\sigma), \frac{d}{d\sigma} \begin{bmatrix} \psi \\ l \end{bmatrix}(\sigma) \right\rangle_{\frac{1}{2},-\frac{1}{2}} d\sigma + \int_0^t \int_{-1}^1 v_y(\sigma,y)\psi_y(\sigma,y)\,\mathrm{d}y\,\mathrm{d}\sigma$$
$$- \frac{1}{2}\int_0^t \int_{-1}^1 v^2(\sigma,y)\psi_y(\sigma,y)\,\mathrm{d}y\,\mathrm{d}\sigma = \int_0^t (-k_v\dot{h}(\sigma) + k_p(h_1 - h(\sigma)))l(\sigma)\,\mathrm{d}\sigma, \quad (5.5)$$

for every $\begin{bmatrix} \psi \\ l \end{bmatrix} \in \mathcal{C}([0,T];H) \cap L^2([0,T];H_{\frac{1}{2}}) \cap \mathcal{H}^1((0,T),H_{-\frac{1}{2}})$.

Now, we take $\begin{bmatrix} \psi \\ l \end{bmatrix} = \begin{bmatrix} v \\ g \end{bmatrix}$ in (5.5), we obtain that

$$\frac{1}{2}\left\| \begin{bmatrix} v(t) \\ g(t) \end{bmatrix} \right\|^2 - \frac{1}{2}\left\| \begin{bmatrix} v_0 \\ g_0 \end{bmatrix} \right\|^2 + \int_0^t \int_{-1}^1 v_y^2 dy d\sigma - \frac{1}{2}\int_0^t \int_{-1}^1 v^2(\sigma,y)v_y(\sigma,y)\,\mathrm{d}y\,\mathrm{d}\sigma$$
$$= \int_0^t (-k_v\dot{h}(\sigma) + k_p(h_1 - h(\sigma)))\dot{h}(\sigma)\,\mathrm{d}\sigma. \quad (5.6)$$

Using the obvious facts that

$$\int_0^t \int_{-1}^1 v^2 v_y\,\mathrm{d}y\,\mathrm{d}\sigma = 0,$$

and

$$\int_0^t (-k_v\dot{h}(\sigma) + k_p(h_1 - h(\sigma)))\dot{h}(\sigma)\,\mathrm{d}\sigma = \int_0^t k_v g^2(\sigma) d\sigma - \frac{k_p}{2}\left((h_1 - h(t))^2 - (h_1 - h_0)^2\right),$$

together with (5.6), it follows that (5.4) holds. □

We can now pass to prove the main result of this section.

Proof of Theorem 5.1. The solution constructed in Theorem 4.1 can be extended to a maximal one defined on the interval $[0,T_{max})$, with $T_{max} \in (0,\infty]$. Energy estimate (5.4) and (5.2) imply that

$$\int_{-1}^1 v^2(t,y)\mathrm{d}y + g^2(t) \leqslant 2k_p|h_0 - h_1|^2 \qquad (t \in [0,T_{max})) \quad (5.7)$$
$$\frac{k_p}{2}|h(t) - h_1|^2 \leqslant k_p|h_0 - h_1|^2 \qquad (t \in [0,T_{max})),$$

so that

$$|h(t) - h_1| \leqslant \sqrt{2}|h_0 - h_1| \qquad (t \in [0,T_{max})).$$

The last formula and (5.1) imply that

$$|h(t) - h_1| \leqslant \frac{1}{2}\min(1 - h_1, 1 + h_1) \qquad (t \in [0,T_{max})),$$

which clearly yields that

$$\min(1 - h(t), 1 + h(t)) \geqslant \frac{1}{2}\min(1 - h_1, 1 + h_1) \qquad (t \in [0,T_{max})). \quad (5.8)$$

Let $\kappa = 2k_p|h_0 - h_1|^2$ and $\varepsilon = \frac{1}{8}\min(1 - h_1, 1 + h_1)$. From (5.7) and (5.8) it follows that we can apply Theorem 4.1 to obtain the existence of $\varrho > 0$ depending only on v_0, g_0, h_0, h_1 such that for every $t \in [0, T_{max})$ the solution of (1.1) can be extended to a finite energy solution defined on $[0, t+\varrho]$. Consequently we have $T_{max} = \infty$ and estimate (5.3) holds true.

Finally, the continuity property stated at the end of the theorem follows by repetitively applying the continuity of the map defined in (4.5) on the intervals $[(n-1)\varrho, n\varrho]$, with $n \in \mathbb{N}$.

$\square$

6 Large time behaviour of the closed loop system

Once we have proved the existence of the global solution of (1.1), we pass to study its asymptotic behavior for $t \to \infty$. The main result of this section reads as follows:

Theorem 6.1. *Under the assumptions of Theorem* 5.1, *the finite energy solution of* (1.1) *satisfies*

$$\lim_{t\to\infty} \|v(t,\cdot)\|_{L^2[-1,1]} = 0, \quad \lim_{t\to\infty} g(t) = 0, \quad \lim_{t\to\infty} h(t) = h_1.$$

In our proof of the above theorem we need the functions $W_1,\ W_2 : L^2[-1,1]\times\mathbb{R}\times[-1,1] \to [0,\infty)$ defined by

$$W_1 \begin{bmatrix} \varphi \\ g \\ h \end{bmatrix} = \frac{1}{2}\left(\int_{-1}^{1} \varphi^2 \,\mathrm{d}y + |g|^2\right) \qquad (\varphi \in L^2[-1,1],\ g \in \mathbb{R},\ h \in [-1,1]), \tag{6.1}$$

$$W_2 \begin{bmatrix} \varphi \\ g \\ h \end{bmatrix} = \frac{k_p}{2}|h-h_1|^2 \qquad (\varphi \in L^2[-1,1],\ g \in \mathbb{R},\ h \in [-1,1]). \tag{6.2}$$

Moreover, we introduce the map $D : \mathcal{H}_0^1(-1,1)\times\mathbb{R}\times[-1,1] \to [0,\infty)$ defined by

$$D \begin{bmatrix} \varphi \\ g \\ h \end{bmatrix} = \int_{-1}^{1} \varphi_y^2(y)\,\mathrm{d}y + k_v g^2 \qquad (\varphi \in \mathcal{H}_0^1(-1,1),\ g \in \mathbb{R},\ h \in [-1,1]). \tag{6.3}$$

On the other hand, given $v_0 \in L^2[-1,1]$, $g_0 \in \mathbb{R}$ and $h_0 \in (-1,1)$, we set

$$S(t) \begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix} = \begin{bmatrix} v(t,\cdot) \\ g(t) \\ h(t) \end{bmatrix} \qquad (t \geqslant 0), \tag{6.4}$$

where $\begin{bmatrix} v \\ h \\ \dot{h} \end{bmatrix}$ is the corresponding solution of (1.1) constructed in Theorem 5.1. The last part of the statement of Theorem 5.1 can be rephrased to say that

$$S(t) \in C\left(L^2[-1,1]\times\mathbb{R}\times(-1,1); L^2[-1,1]\times\mathbb{R}\times(-1,1)\right) \qquad (t \geqslant 0), \tag{6.5}$$

and that, for every $T > 0$, the map

$$\begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix} \mapsto D\left(S(\cdot)\begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix}\right) \tag{6.6}$$

is continuous from $L^2[-1,1]\times\mathbb{R}\times(-1,1)$ to $L^1[0,T]$.

With the above notation, estimate (5.4) writes, for every $t \geqslant 0$,

$$W_1 \begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix} + W_2 \begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix} - W_1 \left(S(t) \begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix} \right) - W_2 \left(S(t) \begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix} \right) = \int_0^t D \left(S(\sigma) \begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix} \right) \mathrm{d}\sigma. \quad (6.7)$$

Proposition 6.2. *Under the assumptions of Theorem 5.1, for every $v_0 \in L^2[-1,1]$, $g_0 \in \mathbb{R}$ and $h_0 \in (-1,1)$ we have that*

$$\lim_{t \to \infty} W_1 \left(S(t) \begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix} \right) = 0.$$

Proof. Within this proof we denote, for the sake of simplicity

$$W_k \left(S(t) \begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix} \right) = W_k(t) \qquad (k \in \{1,2\}, \quad t \geqslant 0).$$

Let us assume, by contradiction, that W_1 does not converge to zero for $t \to \infty$. This means that there exists $\varepsilon > 0$ and a sequence $(t_n)_{n \geqslant 0}$ of positive numbers such that $t_n \to \infty$ and

$$W_1(t_n) \geqslant \varepsilon \qquad (n \in \mathbb{N}).$$

Denote

$$\delta_n = \max \left\{ \delta > 0 \;\mid\; W_1(t_n - \delta) \geqslant \frac{\varepsilon}{2} \right\} \qquad (n \in \mathbb{N}).$$

Since, according to (6.7) and to Poincaré's inequality, we have that $W_1 \in L^1[0,\infty)$, it follows that

$$\sum_{n \in \mathbb{N}} \delta_n < \infty,$$

so that

$$\lim_{n \to \infty} \delta_n = 0.$$

On the other hand, (6.7) implies that $W_1 + W_2$ is nonincreasing so that

$$\begin{aligned} \frac{\varepsilon}{2} + \frac{1}{2}|h(t_n - \delta_n) - h_1|^2 &= W_1(t_n - \delta_n) + W_2(t_n - \delta_n) \\ &\geqslant W_1(t_n) + W_2(t_n) \geqslant \varepsilon + \frac{1}{2}|h(t_n) - h_1|^2 \qquad (n \in \mathbb{N}), \end{aligned}$$

so that

$$|h(t_n - \delta_n) - h_1|^2 - |h(t_n) - h_1|^2 \geqslant \varepsilon \qquad (n \in \mathbb{N}).$$

By applying the mean value theorem and the fact that

$$|h(t) - h_1| \leqslant 2 \qquad (t \geqslant 0),$$

it follows that for every $n \in \mathbb{N}$ there exist $\alpha_n \in (0,1)$ such that

$$|\dot{h}(t_n - \alpha_n \delta_n)| \geqslant \frac{\varepsilon}{4\delta_n} \to \infty.$$

The above estimate clearly contradicts the fact that $W_1 \in L^1[0,\infty)$. $\square$

Now, we are able to prove the main result of this section.

Proof of Theorem 6.1. We know from Proposition 6.2 that

$$\lim_{t\to\infty} \|v(t,\cdot)\|_{L^2[-1,1]} = 0, \qquad \lim_{t\to\infty} \dot{h}(t) = 0. \tag{6.8}$$

Moreover, since $h(t) \in [-1+\varepsilon, 1-\varepsilon]$ for every $t \geqslant 0$ we have that the set $(h(t))_{t\geqslant 0}$ is relatively compact in $\mathbb{R}$. Let $(t_n)_{n\geqslant 0}$ be a sequence of positive numbers such that

$$t_n \to \infty, \qquad \lim_{n\to\infty} h(t_n) = h^* \in [-1+\varepsilon, 1-\varepsilon]. \tag{6.9}$$

We also know from (6.7) that the map $t \mapsto D\left(S(t)\begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix}\right)$ is in $L^1[0,\infty)$ so that, given $T > 0$, we have

$$\lim_{n\to\infty} \int_{t_n}^{T+t_n} D\left(S(t)\begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix}\right) \mathrm{d}t = 0.$$

A change of variables and the semigroup property of the family $(S(t))_{t\geqslant 0}$ imply that

$$\lim_{n\to\infty} \int_0^T D\left(S(s)S(t_n)\begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix}\right) \mathrm{d}s = 0.$$

On the other hand, we know from Theorem 5.1 that $S(s)$ is continuous on $L^2[-1,1]\times\mathbb{R}\times(-1,1)$, so that we can use (6.8) and (6.9) to obtain

$$\lim_{n\to\infty} S(t_n)\begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ h^* \end{bmatrix}.$$

The last two formulae and the continuity of map defined in (6.6) imply that

$$D\left(S(s)\begin{bmatrix} 0 \\ 0 \\ h^* \end{bmatrix}\right) = 0 \qquad (s \in [0,T]).$$

If, for each $t \geqslant 0$, we set $\begin{bmatrix} \widetilde{v}(t,\cdot) \\ \widetilde{g}(t) \\ \widetilde{h}(t) \end{bmatrix} = S(t)\begin{bmatrix} 0 \\ 0 \\ h^* \end{bmatrix}$, it follows that $\widetilde{v}_y(s,\cdot) = 0$ in $L^2[-1,1]$ for almost every $s \in [0,T]$. Moreover, since $\widetilde{v}$ vanishes for $y = \pm 1$, it follows that $\widetilde{v}(s,\cdot) = 0$ in $\mathcal{H}_0^1(-1,1)$ for almost every $s \in [0,T]$. This implies, in particular, that $g(s) = 0$ for almost every $s \in [0,T]$. Finally, using (5.4) (with $\widetilde{v}$ instead of v and h^* instead of h_0) we obtain that $h^* = h_1$, which ends the proof. $\square$

7 Null controllability of a linearized problem

As mentioned in the Introduction, the last step in the proof of our main result consists in proving that, given $T > 0$, any initial state close enough to a target of the form $\begin{bmatrix} 0 \\ 0 \\ h_1 \end{bmatrix}$ can be steered exactly to this target in time T. To accomplish this goal, it seems convenient to linearize the system around the final state instead of the initial one (as it was the case in proving local in time existence of solutions in Section 4). Consequently, we consider a linear operator A_1, which differs from the operator A_0 introduced in Section 3 just by the fact that h_0 is replaced by h_1. More precisely, denoting $H = L^2[-1,1] \times \mathbb{R}$ and given $h_1 \in (-1,1)$, the operator $A_1 : \mathcal{D}(A_1) \to H$ is defined by

$$\mathcal{D}(A_1) = \left\{ \begin{bmatrix} \varphi \\ p \end{bmatrix} \in \mathcal{H}_0^1(-1,1) \times \mathbb{R} \;\middle|\; \varphi_{|(-1,h_1)} \in \mathcal{H}^2(-1,h_1), \varphi_{|(h_1,1)} \in \mathcal{H}^2(h_1,1),\, \varphi(h_1) = p \right\}, \quad (7.1)$$

$$A_1 \begin{bmatrix} \varphi \\ p \end{bmatrix} = \begin{bmatrix} -\{\varphi_{xx}\}_{h_1} \\ -[\varphi_x]_{h_1} \end{bmatrix} \qquad \left(\begin{bmatrix} \varphi \\ p \end{bmatrix} \in \mathcal{D}(A_1) \right), \quad (7.2)$$

where the notation (namely for $\{\varphi_{xx}\}_{h_1}$ and $[\varphi_{xx}]_{h_1}$) is the same as in Section 3.

Moreover, in the remaining part of this article, we use the change of variable Φ defined in Lemma 2.1, with h_0 replaced by h_1. The spaces $H_{\frac{1}{2}}$ and $H_{-\frac{1}{2}}$ are modified accordingly.

Let $B \in \mathcal{L}(\mathbb{C}, H)$ and $C \in \mathcal{L}(H, \mathbb{C})$ be the operators defined by

$$Bw = \begin{bmatrix} 0 \\ w \end{bmatrix} \qquad (w \in \mathbb{C}), \quad (7.3)$$

and

$$C \begin{bmatrix} \varphi \\ p \end{bmatrix} = p \qquad \left(\begin{bmatrix} \varphi \\ p \end{bmatrix} \in H \right). \quad (7.4)$$

By denoting

$$Y(t) = \begin{bmatrix} z(t) \\ g(t) \end{bmatrix},$$

we introduce the controlled linear system

$$\begin{cases} \dot{Y}(t) + A_1 Y(t) = Bw(t) \\ \dot{h}(t) = CY(t) \\ Y(0) = Y_0 \\ h(0) = h_0, \end{cases} \quad (7.5)$$

where $Y_0 = \begin{bmatrix} z_0 \\ g_0 \end{bmatrix} \in H$ and $h_0 \in (-1,1)$. The state trajectory of this system is $\begin{bmatrix} Y \\ h \end{bmatrix}$ and w is the control function. Using Proposition 3.1 with A_0 replaced by A_1 and the variation of constants formula, we see that for every $Y_0 \in H$, $h_0 \in (-1,1)$ and $w \in L^2(0,\infty)$ the system (7.5) has a unique solution given by

$$Y(t) = \begin{bmatrix} z(t) \\ g(t) \end{bmatrix} = \mathbb{T}_t Y_0 + \int_0^t \mathbb{T}_{t-s} Bw(s)\,\mathrm{d}s, \quad h(t) = h_0 + \int_0^t CY(s)\,\mathrm{d}s, \quad (7.6)$$

where $\mathbb{T}$ is the contraction semigroup generated by $-A_1$.

The aim of this section is to study the following linear controllability problem: given $T > 0$ and $\begin{bmatrix} Y_0 \\ h_0 \end{bmatrix} \in H \times \mathbb{R}$, find a control function $w \in \mathcal{C}[0,T]$ such that

$$Y(T) = 0, \; h(T) = h_1. \quad (7.7)$$

The main result of this section says that this problem admits at least one solution, provided that h_1 lies in a certain class of irrationals or, more precisely, in the set

$$\mathcal{S} = \{ a \in (-1,1) \mid a \text{ is an irrational algebraic number} \}. \quad (7.8)$$

The main result in this section states as follows:

Theorem 7.1. *Let $T > 0$ and $h_1 \in \mathcal{S}$. Then for each $Y_0 \in H$ and $h_0 \in \mathbb{R}$ there exists a control $w \in \mathcal{C}[0, T]$ such that the solution $\begin{bmatrix} Y \\ h \end{bmatrix}$ of* (7.5) *verifies* (7.7) *and*

$$\|w\|_{\mathcal{C}[0,T]} \leqslant \kappa_0 e^{\frac{\kappa_1}{T}} \left(\|Y_0\|_H + |h_1 - h_0|\right), \tag{7.9}$$

where κ_0 and κ_1 are two positive constants independent of T and of the data Y_0 and h_0. The constant κ_1 depends on the distance $\min\{h_1 + 1, 1 - h_1\}$ between h_1 and the extremities of the interval $(-1, 1)$, whereas κ_0 depends on the diophantine approximation properties of h_1.

Remark 7.2. *It is well known that a necessary condition for the controllability of system* (7.5) *is*

$$B^*\Phi_n \neq 0, \tag{7.10}$$

for any eigenfunction Φ_n of the operator A_1. From the proof of Theorem 10.1 we can deduce that, if $h_1 \in \mathbb{Q}$, there exists an eigenfunction Φ_n of A_1 which does not satisfy (7.10). *This shows that, for any $h_1 \in \mathbb{Q}$, system* (7.5) *is not controllable. If $h_1 \notin \mathbb{Q}$ condition* (7.10) *is verified. However, in order to obtain the exact controllability and to give the estimate of the control cost* (7.9) *we need to impose additional conditions on h_1. Indeed, if $h_1 \in \mathcal{S}$, we can bound from bellow the distance from h_1 to all rational numbers and we can obtain our desired cost estimate.*

The first step in proving Theorem 7.1 consists in reducing it to an appropriate moment problem. To state this problem, denote by $(\Phi_n)_{n\geqslant 1}$ an orthonormal basis in H formed of eigenvectors of A_1 and let $(\lambda_n)_{n\geqslant 1}$ be the corresponding sequence of eigenvalues. Also, let $\lambda_0 = 0$. The semigroup $\mathbb{T}$ generated by $-A_1$ writes

$$\mathbb{T}_t Y_0 = \sum_{n\geqslant 1} \langle Y_0, \Phi_n\rangle \mathrm{e}^{-\lambda_n t}\Phi_n \qquad (Y_0 \in H).$$

Consequently, (7.6) becomes

$$\begin{aligned} Y(t) &= \sum_{n\geqslant 1} \left[\langle Y_0, \Phi_n\rangle \mathrm{e}^{-\lambda_n t} + \int_0^t \mathrm{e}^{-\lambda_n (t-\sigma)} w(\sigma) B^*\Phi_n \,\mathrm{d}\sigma \right] \Phi_n, \\ h(t) &= h_0 + \int_0^t CY(s)\,\mathrm{d}s. \end{aligned} \tag{7.11}$$

From the above formulas, using standard calculations, we can easily prove the following result:

Proposition 7.3. *Let $T > 0$, $h_1 \in (-1, 1)$, $Y_0 \in H$ and $h_0 \in \mathbb{R}$. Then $w \in L^2[0, T]$ is a control which leads the solution $\begin{bmatrix} Y \\ h \end{bmatrix}$ of* (7.5) *to verify* (7.7) *if and only if*

$$\begin{cases} B_0^*\Phi_n \displaystyle\int_0^T e^{s\lambda_n} w(s)\,\mathrm{d}s & = \quad -\langle Y_0, \Phi_n\rangle \qquad (n \geqslant 1) \\ \left(\displaystyle\sum_{n\in\mathbb{N}^*} \frac{1}{\lambda_n} C\Phi_n B^*\Phi_n\right) \displaystyle\int_0^T w(s)\,\mathrm{d}s & = \quad h_1 - h_0 - \displaystyle\sum_{n\in\mathbb{N}^*} \frac{1}{\lambda_n} C\Phi_n \langle Y_0, \Phi_n\rangle. \end{cases} \tag{7.12}$$

In the sequel K will denote a positive constant which may change from one line to another but it will always be independent of other parameters of the problem. We are now in a position to prove the main result of this section.

Proof of Theorem 7.1. In this proof we make extensive use of the results from Appendixes A and B. Let $(F_n)_{n\geqslant 0}$ be the biorthogonal family to $\left(e^{\lambda_n t}\right)_{n\geqslant 0}$ in $L^2\left[-\frac{T}{2}, \frac{T}{2}\right]$ constructed in Corollary 11.3 from Appendix B. We set

$$w(t) = a_0 F_0(t + T/2) - \sum_{n\geqslant 1} \frac{\langle Y_0, \Phi_n\rangle}{B^*\Phi_n} e^{-\lambda_n T/2} F_n(t + T/2) \qquad (t \in [0, T]), \tag{7.13}$$

where

$$a_0 = \frac{h_1 - h_0 - \sum_{n\geqslant 1} \frac{C\Phi_n}{\lambda_n} \langle Y_0, \Phi_n\rangle}{\sum_{n\geqslant 1} \frac{C\Phi_n B^*\Phi_n}{\lambda_n}}. \tag{7.14}$$

The fact that (7.13) defines a function from $\mathcal{C}[0,T]$ follows from the absolute convergence of the series from the right hand side member. To show this, firstly note that from Corollary 11.3 we have

$$|a_0|\, \|F_0\|_{\mathcal{C}[-\frac{T}{2},\frac{T}{2}]} + \sum_{n\geqslant 1} \left|\frac{\langle Y_0, \Phi_n\rangle}{B^*\Phi_n}\right| e^{-\lambda_n T/2} \|F_n\|_{\mathcal{C}[-\frac{T}{2},\frac{T}{2}]}$$
$$\leqslant c e^{\frac{\kappa}{T}} \left[|a_0| + \sum_{n\geqslant 1} \left|\frac{\langle Y_0, \Phi_n\rangle}{\lambda_n\, B^*\Phi_n}\right| e^{-\lambda_n T/2 + \omega\sqrt{\lambda_n}} \right], \tag{7.15}$$

where c, κ and ω are the constants from (11.20). This means, in particular, that all these constants depend only on the distance $\min\{1+h_1, 1-h_1\}$ (see Remark 11.4). We remark that, according to (10.4) from Appendix A, we have

$$B^*\Phi_n = C\Phi_n = \frac{1}{\sqrt{D(\lambda_n)}} \qquad (n \geqslant 1), \tag{7.16}$$

where $D(\lambda_n)$ is defined in (10.5) from Appendix A. From (7.15), (7.16) and the Cauchy-Schwarz inequality we deduce that

$$|a_0|\, \|F_0\|_{\mathcal{C}[-\frac{T}{2},\frac{T}{2}]} + \sum_{n\geqslant 1} \left|\frac{\langle Y_0, \Phi_n\rangle}{B^*\Phi_n}\right| e^{-\lambda_n T/2} \|F_n\|_{\mathcal{C}[-\frac{T}{2},\frac{T}{2}]}$$
$$\leqslant \tilde{c} e^{\frac{2\kappa}{T}} \left[|a_0|^2 + \sum_{n\geqslant 1} D(\lambda_n)\, |\langle Y_0, \Phi_n\rangle|^2\, e^{-\lambda_n T + 2\omega\sqrt{\lambda_n}} \right]$$
$$\leqslant \tilde{c} e^{\frac{2\kappa}{T}} \left[|a_0|^2 + \sum_{\substack{n\geqslant 1\\ \lambda_n \leqslant \frac{9\omega^2}{T^2}}} D(\lambda_n)\, |\langle Y_0, \Phi_n\rangle|^2\, e^{\frac{6\omega^2}{T}} + \sum_{\substack{n\geqslant 1\\ \lambda_n > \frac{9\omega^2}{T^2}}} D(\lambda_n)\, |\langle Y_0, \Phi_n\rangle|^2\, e^{-\frac{\lambda_n T}{3}} \right]. \tag{7.17}$$

Note that $\tilde{c} = c\left(1 + \sum_{n\geqslant 1} \frac{1}{\lambda_n^2}\right)$ and consequently it depends only on $\min\{1+h_1, 1-h_1\}$. From (7.14), (7.16), the Cauchy-Schwarz inequality and the obvious fact, following from(10.5), that $D(\lambda_n) \geqslant \frac{1}{2}$, we obtain that there exists a positive constant K, depending only on $\min\{1+h_1, 1-h_1\}$, such that

$$|a_0|^2 \leqslant K \left(|h_1 - h_0|^2 + \sum_{n\geqslant 1} |\langle Y_0, \Phi_n\rangle|^2 \right). \tag{7.18}$$

To evaluate of the last two terms in (7.17), we remark that, for any $T > 0$, we have

$$D(\lambda_n) \leqslant \begin{cases} \frac{\widetilde{M}}{T^4} & \text{if } \lambda_n \leqslant \frac{9\omega^2}{T^2} \\ \frac{\widetilde{M}}{T^2} e^{\frac{\lambda_n T}{3}} & \text{if } \lambda_n > \frac{9\omega^2}{T^2}, \end{cases} \tag{7.19}$$

where $\widetilde{M}$ depends on the distance $\min\{1+h_1, 1-h_1\}$ and on the diophantine approximation properties of h_1. Indeed, using the fact that $h_1 \in \mathcal{S}$, we can apply Lemma 10.6 from Appendix A with $\varsigma = 1$ and we deduce that

$$D(\lambda_n) \leqslant M n^4,$$

where M is the constant from (10.18) which depends on the distance $\min\{1+h_1, 1-h_1\}$ and on the diophantine approximation properties of h_1. By taking into account the properties of λ_n from Theorem 10.5, we immediately obtain that (7.19) holds. From (7.17), (7.18) and (7.19) we deduce immediately that

$$|a_0| \, \|F_0\|_{\mathcal{C}[-\frac{T}{2},\frac{T}{2}]} + \sum_{n\geqslant 1} \left| \frac{\langle Y_0, \Phi_n\rangle}{B^*\Phi_n} \right| e^{-\lambda_n T/2} \|F_n\|_{\mathcal{C}[-\frac{T}{2},\frac{T}{2}]}$$

$$\leqslant \kappa_0 \max\left\{\frac{1}{T^4}, \frac{1}{T^2}\right\} e^{\frac{2\kappa+6\omega^2}{T}} \left(|h_1 - h_0|^2 + \sum_{n\geqslant 1} |\langle Y_0, \Phi_n\rangle|^2 \right),$$

which implies that the right hand side of (7.13) is absolutely convergent and gives a function $w \in \mathcal{C}[0,T]$ verifying (7.9). The constant κ_0 is equal to $\max\left\{\tilde{c}K, \tilde{c}\widetilde{M}\right\}$ and thus depends on the distance $\min\{1+h_1, 1-h_1\}$ and on the diophantine approximation properties of h_1. By choosing κ_1 any real number greater than $2\kappa + 6\omega^2$, we deduce that (7.9) holds. According to the properties of the biorthogonal sequence $(F_n)_{n\geqslant 0}$, we have that w is a solution of the moment problem (7.12), so that, by Proposition 7.3, w steers the solution of (7.5) to a state satisfying (7.7). $\square$

Remark 7.4. *As shown in Theorem 10.5 from Appendix A, the sequence of eigenvalues $(\lambda_m)_{m\geqslant 1}$ of the operator A_1 is the union of two increasing subsequences of positive numbers, $(\lambda^1_n)_{n\geqslant 1}$ and $(\lambda^2_k)_{k\geqslant 1}$. The presence of two families of exponents can be encountered in the controllability theory for underactuated parabolic systems (see, for instance, Ammar Khodja et al. [2]). However, the situation in this paper differs from the one described in [2], since our two families are not exponentially close. Indeed, as shown by property* (10.17) *in Theorem 10.5, there is a positive gap between the families $(\lambda^1_n)_{n\geqslant 1}$ and $(\lambda^2_k)_{k\geqslant 1}$ which, consequently, verify hypothesis (Λ_3) from Appendix B. Therefore, unlike in [2], we have null controllability in arbitrarily small time.*

8 Adding a source term

In this section we consider a control system derived from (7.5) by adding appropriate source terms. This system is defined by:

$$\begin{cases} \dot{Y}(t) + A_1 Y(t) = Bw(t) + f(t) \\ \dot{h}(t) = CY(t) \\ Y(0) = Y_0 \\ h(0) = h_0, \end{cases} \tag{8.1}$$

where we have used the same notation as in Section 7 for A_1, B and C. Before stating the controllability result for (8.1), we need more notation. As in Section 7, let $h_1 \in \mathcal{S}$ and let $\gamma : (0,\infty) \to (0,\infty)$ be the cost function appearing in Theorem 7.1, i.e.

$$\gamma(t) = \kappa_0 e^{\frac{\kappa_1}{t}} \qquad (t > 0),$$

where κ_0 and κ_1 are the constants in (7.9). Moreover, given $\tau > 0$, we consider the functions

$$\rho_{\mathcal{F}}(t) = e^{-\frac{\alpha}{(\tau-t)^2}}, \quad \rho_0(t) = \kappa_0 e^{\frac{\kappa_1}{(q-1)(\tau-t)} - \frac{\alpha}{q^4(\tau-t)^2}} \qquad (t \in [0,\tau)), \tag{8.2}$$

where $q > 1$ and

$$\alpha > \frac{\kappa_1 q^4 \tau}{2(q-1)}.$$

Note that, thanks to the choice of α, these functions can be extended by continuity for $t = \tau$, with $\rho_{\mathcal{F}}(\tau) = \rho_0(\tau) = 0$.

To these functions we associate the following Hilbert spaces

$$\mathcal{F} = \left\{ f \in L^2([0,\tau]; H_{-\frac{1}{2}}) \;\middle|\; \frac{f}{\rho_{\mathcal{F}}} \in L^2([0,\tau]; H_{-\frac{1}{2}}) \right\}, \tag{8.3}$$

$$\mathcal{W} = \left\{ w \in L^2[0,\tau] \;\middle|\; \frac{w}{\rho_0} \in L^2[0,\tau] \right\}, \tag{8.4}$$

$$\mathcal{Z} = \left\{ z \in L^2([0,\tau]; H) \;\middle|\; \frac{z}{\rho_0} \in L^2([0,\tau]; H) \right\}. \tag{8.5}$$

The inner product in $\mathcal{F}$ is defined by

$$\left\langle f, \widetilde{f} \right\rangle_{\mathcal{F}} = \int_0^\tau \rho_{\mathcal{F}}^{-2}(t) \left\langle f(t), \widetilde{f}(t) \right\rangle_{-\frac{1}{2}} \mathrm{d}t \qquad (f, \widetilde{f} \in \mathcal{F}),$$

and similar definitions are considered in $\mathcal{W}$ and $\mathcal{Z}$. The induced norms are denoted by $\|\cdot\|_{\mathcal{F}}$, $\|\cdot\|_{\mathcal{W}}$ and $\|\cdot\|_{\mathcal{Z}}$, respectively. We recall that the inner product $\langle\ ,\ \rangle_{-\frac{1}{2}}$ is defined by

$$\left\langle \xi, \widetilde{\xi} \right\rangle_{-\frac{1}{2}} = \left\langle A_1^{-1}\xi, A_1^{-1}\widetilde{\xi} \right\rangle_{\frac{1}{2}} \qquad (\xi, \widetilde{\xi} \in H_{-\frac{1}{2}}).$$

The main result of this section can be seen as a version of Proposition 2.3 in [19] and states as follows. Therefore, we state it below and we omit its proof.

Theorem 8.1. *With the above notation and assumption, let $\tau > 0$ and $f \in \mathcal{F}$. Then, for any $\begin{bmatrix} Y_0 \\ h_0 \end{bmatrix} \in H \times \mathbb{R}$ and $h_1 \in \mathcal{S}$, there exists $w \in \mathcal{W} \cap \mathcal{C}[0,\tau]$ such that the solution $\begin{bmatrix} Y \\ h \end{bmatrix}$ of (8.1) satisfies $Y \in \mathcal{Z}$ and $h(\tau) = h_1$. Moreover, there exists a positive constant K, not depending on f, Y_0 and h_0 (but it may depend on h_1 and on τ) such that*

$$\left\| \frac{Y}{\rho_0} \right\|_{\mathcal{C}([0,\tau],H)} + \left\| \frac{h - h_1}{\rho_0} \right\|_{\mathcal{C}[0,\tau]} + \left\| \frac{w}{\rho_0} \right\|_{\mathcal{C}[0,\tau]} \leqslant K \left(\|f\|_{\mathcal{F}} + \|Y_0\| + |h_0 - h_1| \right). \tag{8.6}$$

Remark 8.2. *According to Theorem 8.1, given $\tau > 0$ and $h_1 \in \mathcal{S}$, there exists a map*

$$E_\tau : H \times \mathbb{R} \times \mathcal{F} \to \mathcal{W}$$

such that, for any $(Y_0, h_0, f) \in H \times \mathbb{R} \times \mathcal{F}$, the control $w = E_\tau(Y_0, h_0, f) \in \mathcal{W}$ is such that the solution Y of (8.1) verifies $Y \in \mathcal{Z}$ and $h(\tau) = h_1$. Moreover, the following estimate holds

$$\|E_\tau(Y_0, h_0, f)\|_{\mathcal{W}} \leqslant K \left(\|f\|_{\mathcal{F}} + \|Y_0\| + |h_0 - h_1| \right), \tag{8.7}$$

where $K > 0$ is a constant independent of f, Y_0 and h_0 (it may depend of h_1 and τ).

Moreover, from the proof of Theorem 8.1, we easily deduce that

$$\|E_\tau(Y_0, h_0, f) - E_\tau(Y_0, h_0, \tilde{f})\|_{\mathcal{W}} \leqslant K \|f - \tilde{f}\|_{\mathcal{F}}, \tag{8.8}$$

where, once more, $K > 0$ is a constant independent of Y_0, h_0, f and $\tilde{f}$.

Corollary 8.3. *With the assumptions and notation from Theorem 8.1, denote*

$$\rho(t) = e^{-\frac{\beta}{(\tau - t)^2}} \qquad (t \in [0,\tau)), \tag{8.9}$$

where the positive constant β is chosen such that $\beta < \frac{\alpha}{q^4}$.

Then, for every $f \in \mathcal{F}$, $(Y_0, h_0, h_1) \in H \times \mathbb{R} \times \mathcal{S}$, the trajectory Y obtained by solving (8.1) with the control $w \in \mathcal{W}$ given by Theorem 8.1 satisfies $Y \in L^2([0,\tau]; H_{\frac{1}{2}})$. Moreover, there exists a positive constant K, not depending on f, Y_0, h_0 such that

$$\left\| \frac{Y}{\rho} \right\|_{L^2([0,\tau]; H_{\frac{1}{2}})} \leqslant K \left(\|f\|_{\mathcal{F}} + \|Y_0\| + |h_0 - h_1| \right). \tag{8.10}$$

Moreover, assuming that $\beta > \frac{\alpha}{2}$ and $q^4 < 2$, it follows that $\frac{\rho^2}{\rho_{\mathcal{F}}} \in \mathcal{C}[0,\tau]$.

Proof. Let $w \in \mathcal{W}$ be the control constructed in the proof of Theorem 8.1 and Y be the corresponding trajectory. Then $Z = \frac{Y}{\rho}$ satisfies

$$\dot{Z} = A_1 Z + B\frac{w}{\rho} + \frac{f}{\rho} + \frac{\dot{\rho}}{\rho^2}Y. \tag{8.11}$$

Since $\frac{\rho_{\mathcal{F}}}{\rho} \in L^\infty[0,\tau]$, $\frac{\rho_0}{\rho} \in L^\infty[0,\tau]$, $\frac{\dot{\rho}\rho_0}{\rho^2} \in L^\infty[0,\tau]$ then

$$B\frac{w}{\rho} + \frac{f}{\rho} + \frac{\dot{\rho}}{\rho^2}Y \in L^2([0,\tau]; H_{-\frac{1}{2}})$$

and the result follows from classical results (see Lions and Magenes [18, Section 3.4] or Wloka [30]). □

9 Proof of the main result

To prove Theorem 1.2 we first show that the nonlinear system (1.1) is locally exactly controllable to the equilibrium states $\begin{bmatrix} 0 \\ 0 \\ h_1 \end{bmatrix}$, with $h_1 \in \mathcal{S}$, where $\mathcal{S}$ has been defined in (7.8).

Theorem 9.1. *Let $\tau > 0$ and $h_1 \in \mathcal{S}$. Then there exists $\delta > 0$ such that for every $\begin{bmatrix} v_0 \\ g_0 \end{bmatrix} \in H$ and $h_0 \in (-1,1)$ satisfying*

$$\|v_0\|^2_{L^2[-1,1]} + |g_0|^2 \leqslant \delta^2, \quad |h_0 - h_1| < \delta, \tag{9.1}$$

there exists a control $w \in \mathcal{C}[0,\tau]$ such that the solution of the nonlinear system (1.1) verifies

$$v(\tau) = 0, \;\; g(\tau) = 0, \;\; h(\tau) = h_1. \tag{9.2}$$

Proof. Let $\tau > 0$ and let $\rho_{\mathcal{F}}$, ρ_0 and ρ be the weight functions introduced by (8.2) and (8.9), respectively, supposed to satisfy the assumptions in Corollary 8.3. Also, let $\mathcal{F}$, $\mathcal{W}$ and $\mathcal{Z}$ be the functional spaces defined by (8.3)-(8.5). Let $\delta > 0$ to be chosen later on. We set

$$\mathcal{X}_{\tau,\delta} = \left\{ \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in \mathcal{F} \;\middle|\; \left\| \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right\|_{\mathcal{F}} \leqslant \delta \right\}.$$

Given $h_1 \in \mathcal{S}$, let $\varepsilon > 0$ be such that $|h_1| \leqslant 1 - 4\varepsilon$. For $\begin{bmatrix} v_0 \\ g_0 \\ h_0 \end{bmatrix} \in \mathcal{B}_{\delta,\varepsilon}$, we define the operator $\widetilde{\mathcal{N}} : \mathcal{X}_{\tau,\delta} \to L^2([0,\tau]; H_{-\frac{1}{2}})$ by

$$\widetilde{\mathcal{N}} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \mathcal{G}_1 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \mathcal{G}_2 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \mathcal{G}_3 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}. \tag{9.3}$$

In the definition of $\widetilde{\mathcal{N}}$ we have used the notation from Theorem 4.1, in particular for $\mathcal{B}_{\delta,\varepsilon}$ and for the operators $(\mathcal{G}_k)_{1\leqslant k\leqslant 3}$ (introduced in (4.6)-(4.8)). These operators are slightly modified, in the sense that, in their definition, $\begin{bmatrix} z \\ g \\ h \end{bmatrix}$ represents now the controlled solution of (8.1) with initial data $\begin{bmatrix} z_0 \\ g_0 \\ h_0 \end{bmatrix} \in H \times (-1,1)$, where $z_0 = v_0(\Phi(\,\cdot\,,h_0))\Phi_x(\,\cdot\,,h_0)$, and nonhomogeneous term $\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in \mathcal{X}_{\tau,\delta}$. We recall that, in this section, we use the change of variable Φ defined in Lemma 2.1, with h_0

replaced by h_1. If $w = E_\tau\left(\begin{bmatrix} z_0 \\ g_0 \end{bmatrix}, h_0, \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}\right) \in \mathcal{W} \cap \mathcal{C}[0,\tau]$ is the corresponding control given by Theorem 8.1 (see, also, Remark 8.2), it follows that $\begin{bmatrix} z \\ g \end{bmatrix}$ satisfies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1} z(t,x)\varphi(t,x)\,\mathrm{d}x - \langle\dot{\varphi}(t,\cdot), z(t,\cdot)\rangle_{\mathcal{H}^{-1}(-1,1),\mathcal{H}_0^1(-1,1)} + \frac{\mathrm{d}}{\mathrm{d}t}(g(t)l(t)) - g(t)\dot{l}(t) \\ + \int_{-1}^{1} z_x(t,x)\varphi_x(t,x)\,\mathrm{d}x = w(t)l(t) + \left\langle \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}, \begin{bmatrix} \varphi(t) \\ l(t) \end{bmatrix} \right\rangle_{\frac{1}{2},-\frac{1}{2}},\end{aligned} \tag{9.4}$$

for every $\begin{bmatrix} \varphi \\ l \end{bmatrix} \in L^2([0,T]; H_{\frac{1}{2}}) \cap \mathcal{H}^1((0,T); H_{-\frac{1}{2}})$ and for almost every $t \in [0,\tau]$.

The remaining part of this proof follows a classical idea used to study the controllability properties of nonlinear systems: we will show that there exists $\delta > 0$ such that the operator $\widetilde{\mathcal{N}}$ is a contraction on $\mathcal{X}_{\tau,\delta}$. This ensures that $\widetilde{\mathcal{N}}$ has a unique fixed point which, according to (9.4) and Corollary 2.4, gives a solution of (1.1) verifying (9.4).

Hence, to conclude the proof of the theorem, it remains to verify that there exists $\delta > 0$ such that we have

$$\widetilde{\mathcal{N}}(\mathcal{X}_{\tau,\delta}) \subseteq \mathcal{X}_{\tau,\delta} \tag{9.5}$$

and $\widetilde{\mathcal{N}}$ is a contraction in $\mathcal{X}_{\tau,\delta}$.

Proceeding as in the proof of Lemma 4.2 and using the facts that $\frac{\rho_0^2}{\rho_{\mathcal{F}}}$ and $\frac{\rho^2}{\rho_{\mathcal{F}}}$ are in $C[0,\tau]$ (this follows from the definition of these functions and from Corollary 8.3), we obtain that there exists a constant $K(\varepsilon)$ such that

$$\begin{aligned}\int_0^\tau \frac{1}{\rho_{\mathcal{F}}^2(t)} \left\| \mathcal{G}_1 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}(t) \right\|_{-\frac{1}{2}}^2 \mathrm{d}t &\leqslant K(\varepsilon) \int_0^\tau \frac{\rho_0^2(t)\rho^2(t)}{\rho_{\mathcal{F}}^2(t)} \left(\frac{h(t)-h_1}{\rho_0(t)}\right)^2 \int_{-1}^{1} \left|\frac{z_x(t,x)}{\rho(t)}\right|^2 \mathrm{d}x\,\mathrm{d}t \\ &\leqslant K(\varepsilon) \left\| \frac{h-h_1}{\rho_0} \right\|_{\mathcal{C}[0,\tau]}^2 \int_0^\tau \int_{-1}^{1} \left|\frac{z_x(t,x)}{\rho(t)}\right|^2 \mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

From the last inequality, (8.6) and (8.10) we obtain that there exists a constant $K(\varepsilon,\tau)$ depending only on ε and τ such that we have

$$\int_0^\tau \frac{1}{\rho_{\mathcal{F}}^2(t)} \left\| \mathcal{G}_1 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}(t) \right\|_{-\frac{1}{2}}^2 \mathrm{d}t \leqslant K(\varepsilon,\tau) \left(\left\| \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right\|_{\mathcal{F}}^2 + \|Y_0\|^2 + |h_0 - h_1|^2 \right)^2.$$

Similar estimates hold for the operators $\mathcal{G}_2$ and $\mathcal{G}_3$. Taking into account (8.7) and (9.1), we deduce that there exists $\delta > 0$, depending only on h_1 and on τ, such that we have

$$\left\| \mathcal{G}_k \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right\|_{\mathcal{F}} \leqslant K(\varepsilon,\tau)\delta^2 \leqslant \frac{\delta}{3}. \qquad \left(\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in \mathcal{X}_{\tau,\delta},\ k \in \{1,2,3\} \right). \tag{9.6}$$

Inclusion (9.5) follows immediately from (9.6) and definition (9.3) of $\widetilde{\mathcal{N}}$.

On the other hand, using again the facts that $\frac{\rho_0^2}{\rho_{\mathcal{F}}}$ and $\frac{\rho^2}{\rho_{\mathcal{F}}}$ are in $C[0,\tau]$, we deduce that there exists a constant $K(\varepsilon)$ such that we have

$$\begin{aligned}&\int_0^\tau \frac{1}{\rho_{\mathcal{F}}^2(t)} \left\| \mathcal{G}_1 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}(t) - \mathcal{G}_1 \begin{bmatrix} \widetilde{f}_1 \\ \widetilde{f}_2 \end{bmatrix}(t) \right\|_{-\frac{1}{2}}^2 \mathrm{d}t \\ &\leqslant K(\varepsilon) \int_0^\tau \int_{-1}^{1} \left(\left|\frac{z_x(x,t)}{\rho(t)}\right|^2 \left|\frac{h(t)-\widetilde{h}(t)}{\rho_0(t)}\right|^2 + \left(\frac{z_x(x,t)-\widetilde{z}_x(x,t)}{\rho(t)}\right)^2 \left(\frac{\widetilde{h}(t)-h_1}{\rho_0(t)}\right)^2 \right) \mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Taking into account the last inequality, (8.6) and (8.10) we obtain that there exists a constant $K(\varepsilon,\tau)$ depending only on ε and τ such that the following inequality is verified

$$\int_0^\tau \frac{1}{\rho_{\mathcal{F}}^2(t)} \left\| \mathcal{G}_1 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}(t) - \mathcal{G}_1 \begin{bmatrix} \widetilde{f}_1 \\ \widetilde{f}_2 \end{bmatrix}(t) \right\|_{-\frac{1}{2}}^2 \mathrm{d}t$$
$$\leqslant K(\varepsilon,\tau) \left(\|Y_0\|^2 + |h_0 - h_1|^2 + \left\| \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right\|_{\mathcal{F}}^2 + \left\| \begin{bmatrix} \widetilde{f}_1 \\ \widetilde{f}_2 \end{bmatrix} \right\|_{\mathcal{F}}^2 \right) \left(\left\| \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} - \begin{bmatrix} \widetilde{f}_1 \\ \widetilde{f}_2 \end{bmatrix} \right\|_{\mathcal{F}}^2 + \|w - \widetilde{w}\|_{\mathcal{W}}^2 \right).$$

By combining the last estimate and (8.8), it follows that there exists a constant $K(\varepsilon,\tau)$ depending only on ε and τ such that we have

$$\left\| \mathcal{G}_1 \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} - \mathcal{G}_1 \begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{bmatrix} \right\|_{\mathcal{F}} \leqslant K(\varepsilon,\tau)\delta \left\| \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} - \begin{bmatrix} \widetilde{f}_1 \\ \widetilde{f}_2 \end{bmatrix} \right\|_{\mathcal{F}} \qquad \left(\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{bmatrix} \in \mathcal{X}_{\tau,\delta} \right). \tag{9.7}$$

Similar estimates hold for the operators $\mathcal{G}_2$ and $\mathcal{G}_3$. Consequently, there exists $\delta > 0$, depending only on h_1 and on τ, such that the operator $\widetilde{\mathcal{N}}$ is a contraction and the proof of the theorem ends. ☐

Now we have all the ingredients needed to prove our main result.

Proof of Theorem 1.2. Since $\mathcal{S}$ is dense in $[-1,1]$, there exists $h_1 \in \mathcal{S}$ such that

$$|h_F - h_1| < \eta. \tag{9.8}$$

For $\tau > 0$, let $\delta > 0$ be the constant given by Theorem 9.1. Without loss of generality, we can assume that $h_1 > h_0$. Let $N \in \mathbb{N}$ be the smallest integer such that

$$\frac{h_1 - h_0}{N} < \min\left\{ \frac{\delta}{2}, \frac{1}{4\sqrt{2}} \min\{1 - h_1, 1 + h_1\} \right\}.$$

For $j \in \{0, 1, \dots, N\}$ we set

$$h_{0,j} = h_0 + j\frac{h_1 - h_0}{N},$$

so that $h_{0,0} = h_0$ and $h_{0,N} = h_1$. From Theorem 6.1 it follows that there exist k_1, $T_1 > 0$ such that the solution $\begin{bmatrix} v^{(1)} \\ g^{(1)} \\ h^{(1)} \end{bmatrix}$ of (1.1) with

$$u(t) := u^{(1)}(t) = k_1(h_{0,1} - h^{(1)}(t)) \qquad (t \in [0, T_1])$$

satisfies

$$\|v^{(1)}(T_1,\cdot)\|_{L^2[-1,1]}^2 + |g^{(1)}(T_1)|^2 \leqslant \frac{\delta^2}{4},$$
$$|h^{(1)}(T_1) - h_{0,1}| \leqslant \min\left\{ \frac{\delta}{2}, \frac{1}{4\sqrt{2}} \min\{1 - h_1, 1 + h_1\} \right\}.$$

Recursively we can construct the sequences $(k_j)_{1\leqslant j\leqslant N}$ and $(T_j)_{1\leqslant j\leqslant N}$ such that the solution $\begin{bmatrix} v^{(j)} \\ g^{(j)} \\ h^{(j)} \end{bmatrix}$ of the first four equations in (1.1) with

$$v^{(j)}(T_{j-1}) = v^{(j-1)}(T_{j-1}), \quad g^{(j)}(T_{j-1}) = g^{(j-1)}(T_{j-1}), \quad h^{(j)}(T_{j-1}) = h^{(j-1)}(T_{j-1}) \quad (1 \leqslant j \leqslant N),$$
$$u(t) := u^{(j)}(t) = k_j(h_{0,j} - h^{(j)}(t)) \qquad (t \in [T_{j-1}, T_j]),$$

satisfies, for every $1 \leqslant j \leqslant N$,

$$\|v^{(j)}(T_j,\cdot)\|^2_{L^2[-1,1]} + |g^{(j)}(T_j)|^2 \leqslant \frac{\delta^2}{4}, \qquad |h^{(j)}(T_j) - h_{0,j}| \leqslant \min\left\{\frac{\delta}{2}, \frac{1}{4\sqrt{2}}\min\{1-h_1, 1+h_1\}\right\}.$$

Setting, for every $j \in \{1,\ldots,N\}$,

$$\widetilde{u}(t) = u^{(j)}(t) \qquad (t \in [T_{j-1}, T_j]), \tag{9.9}$$

it follows that the corresponding solution $\begin{bmatrix} v \\ g \\ h \end{bmatrix}$ of (1.1) satisfies

$$\|v(T_N,\cdot)\|^2_{L^2[-1,1]} + |g(T_N)|^2 \leqslant \frac{\delta^2}{4}, \qquad |h(T_N) - h_1| \leqslant \frac{\delta}{2}.$$

Using the last two estimates we can apply Theorem 9.1 to deduce that, for any $\tau > 0$, there exists a control $w \in \mathcal{C}[0,\tau]$ such that the solution of (1.1) with the initial data $\begin{bmatrix} v(T_N) \\ g(T_N) \\ h(T_N) \end{bmatrix}$ verifies

$$v(\tau) = 0, \; g(\tau) = 0, \; h(\tau) = h_1. \tag{9.10}$$

Let $T = \tau + T_N$ and define $u \in L^2(0,T)$ by

$$u(t) = \begin{cases} \widetilde{u}(t) & t \in [0, T_N] \\ \\ w(t - T_N) & t \in [T_N, T], \end{cases}$$

where $\widetilde{u}$ is given by (9.9). Using (9.8) and (9.10) it follows that (1.5) holds, which ends the proof. $\square$

10 Appendix A: Spectral properties of the operator A_1

This appendix is devoted to the study of the spectral properties of the operator A_1 introduced in Section 7.

Theorem 10.1. *Let $h_1 \in (-1,1) \setminus \mathbb{Q}$ and let A_1 be the operator defined by* (7.1)-(7.2). *Then the eigenvalues of A_1 are simple and can be arranged as an increasing sequence $(\lambda_n)_{n\geqslant 1}$ of positive numbers which coincides with the set containing the square of each root of the equation*

$$\frac{1}{\tan((1-h_1)x)} + \frac{1}{\tan((1+h_1)x)} = x. \tag{10.1}$$

Moreover, there exists a corresponding sequence of eigenvectors $(\phi_n)_{n\geqslant 1}$ which forms an orthonormal basis of $H = L^2[-1,1] \times \mathbb{R}$.

Proof. Denoting by $\Phi = \begin{bmatrix} \varphi \\ p \end{bmatrix}$ a generic element of $\mathcal{D}(A_1)$, we deduce from the definition and the positiveness of A_1 that Φ is an eigenvector of this operator if and only if there exists $\lambda \geqslant 0$ verifying

$$\begin{cases} -\{\varphi_{xx}\}_{h_1} = \lambda\varphi(x) & x \in (-1,1) \\ \varphi(-1) = \varphi(1) = 0 \\ p = \varphi(h_1) = -\dfrac{1}{\lambda}[\varphi_x]_{h_1}. \end{cases} \tag{10.2}$$

From the first two equations we deduce that there exists two constants C_1 and C_2 such that

$$\varphi(x) = \begin{cases} C_1 \sin\left(\sqrt{\lambda}(1+x)\right) & x \in (-1, h_1) \\ C_2 \sin\left(\sqrt{\lambda}(1-x)\right) & x \in (h_1, 1). \end{cases}$$

The continuity of φ in $x = h_1$ and the third condition in (10.2) imply that that λ verifies

$$\sin((1+h_1)\sqrt{\lambda} + (1-h_1)\sqrt{\lambda}) = \sqrt{\lambda}\sin((1-h_1)\sqrt{\lambda})\sin((1+h_1)\sqrt{\lambda}). \tag{10.3}$$

Note that, if $h_1 \in \mathbb{Q}$, then there exist functions φ such that $\varphi(h_1) = 0$. Hence, h_1 would be a nodal point for an eigenfunction of A_1. Since this property is incompatible with the controllability property of our system, we have chosen to study only the case $h_1 \in (-1,1) \setminus \mathbb{Q}$.

By taking into account again the third condition in (10.2) we obtain from (10.3) that $\sqrt{\lambda}$ is a positive root of (10.1). Hence, the eigenvalues of the operator A_1 are all simple and their set coincides with that of the square of each root of equation (10.1).

The corresponding eigenvectors $(\Phi_n)_{n\geqslant 1}$ are given by

$$\Phi_n = \frac{1}{\sqrt{D(\lambda_n)}} \begin{bmatrix} \varphi_n \\ 1 \end{bmatrix} \quad (n \geqslant 1), \tag{10.4}$$

where

$$D(\lambda_n) = \frac{1+h_1}{2\sin^2\left(\sqrt{\lambda_n}(1+h_1)\right)} + \frac{1-h_1}{2\sin^2\left(\sqrt{\lambda_n}(1-h_1)\right)} + \frac{1}{2}, \tag{10.5}$$

and

$$\varphi_n(x) = \begin{cases} \frac{\sin(\sqrt{\lambda_n}(1+x))}{\sin\left(\sqrt{\lambda_n}(1+h_1)\right)} & x \in (-1, h_1) \\ \frac{\sin(\sqrt{\lambda_n}(1-x))}{\sin\left(\sqrt{\lambda_n}(1-h_1)\right)} & x \in (h_1, 1). \end{cases}$$

From the classical theory of self-adjoint operators we deduce easily that $(\Phi_n)_{n\geqslant 1}$ forms an orthogonal basis in H. $\square$

In the remaining part of this section we study the properties of the roots of equation (10.1). Denote $\alpha = \max\left\{\frac{\pi}{1-h_1}, \frac{\pi}{1+h_1}\right\}$, $\beta = \min\left\{\frac{\pi}{1-h_1}, \frac{\pi}{1+h_1}\right\}$ and $N = \left[\frac{\alpha}{\beta}\right]$. For each $k \geqslant 1$ let $n_k = \left[\frac{k\alpha}{\beta}\right] + 1 \in \mathbb{N}$, which verifies

$$(n_k - 1)\beta < k\alpha < n_k\beta. \tag{10.6}$$

We have the following first result.

Lemma 10.2. *Let $h_1 \in (-1,1)\setminus\mathbb{Q}$. Equation* (10.1) *has two families of positive roots $(y_n)_{n\geqslant 1}$ and $(x_k)_{k\geqslant 1}$ which satisfy $y_n \in ((n-1)\beta, n\beta)$ for each $n \geqslant 1$, $y_{n_k} \in (((n_k-1)\beta, k\alpha)$ and $x_k \in (k\alpha, n_k\beta)$ for each $k \geqslant 1$. Moreover, we have*

$$\lim_{n\to\infty} |y_n - (n-1)\beta| = 0, \tag{10.7}$$

$$\lim_{k\to\infty} |x_k - k\alpha| = 0. \tag{10.8}$$

Proof. A simple argument shows that if x tends to infinity in (10.1) then at least one of the quantities $\frac{1}{\tan((1+h_1)x)}$ and $\frac{1}{\tan((1-h_1)x)}$ tends to infinity. Hence, it follows that the roots of (10.1) satisfy (10.7) and (10.8). $\square$

The following result is well known (see, for instance, [14, Exemple 7.6, p. 197]).

Lemma 10.3. *Let $p > 0$. Equation*

$$\tan(x) = \frac{1}{px} \tag{10.9}$$

has a sequence of positive roots $(r_k)_{k \geqslant 1}$ with the property that

$$r_k = k\pi + \frac{1}{pk\pi} + o\left(\frac{1}{k}\right) \qquad (k \to \infty). \tag{10.10}$$

In the following lemma we study the distance between two consecutive roots of (10.1).

Lemma 10.4. *Let $(y_n)_{n \geqslant 1}$ and $(x_k)_{k \geqslant 1}$ be the two families of roots given by Lemma 10.2. There exists $r > 0$, depending only on* $\min\{1 - h_1, 1 + h_1\}$, *such that the following properties hold*

$$y_{n+1} - y_n > r\beta \qquad (n \geqslant 1,\, n \neq n_k,\, k \geqslant 1), \tag{10.11}$$

$$x_k - y_{n_k} > \frac{r}{k} \qquad (k \geqslant 1), \tag{10.12}$$

$$y_{n_k+1} - x_k > \frac{r}{k} \qquad (k \geqslant 1), \tag{10.13}$$

$$y_n - (n-1)\beta \geqslant \frac{r}{n} \qquad (n \geqslant 1,\, n \neq n_k,\, k \geqslant 1). \tag{10.14}$$

Proof. From (10.7) we deduce immediately that (10.11) holds. For each $k \geqslant 1$, let us denote by $y_k^{\pm}$ the unique solution of the equation

$$\frac{1}{\tan((1 \pm h_1)x)} = x$$

belonging to the interval $((k-1)\beta, k\beta)$ and $(k\alpha, (k+1)\alpha)$, respectively. For simplicity, we denote $I_k = ((n_k - 1)\beta, k\alpha)$ and $I_{k+1} = (k\alpha, n_k\beta)$. We analyze the following cases

1. If $|I_k| < |I_{k+1}|/2$, by using (10.7)-(10.8) we deduce that $y_{n_k+1} - x_k > \frac{\beta}{4}$, which gives (10.12). From (10.8), Lemma 10.3 and the fact that

$$\frac{1}{\tan((1-h_1)x)} + \frac{1}{\tan((1+h_1)x)} \geqslant \frac{1}{\tan((1-h_1)x)} \qquad x \in \left(k\alpha, (n_k - 1)\beta + \frac{\beta}{2}\right),$$

we obtain that there exists $r > 0$ such that

$$x_k - y_{n_k} > x_k - k\alpha > y_k^- - k\alpha > \frac{r}{k}.$$

Hence, (10.13) is verified.

2. If $|I_k| \geqslant |I_{k+1}|/2$, by using (10.7)-(10.8) we have that $x_k - y_{n_k} > \frac{\beta}{4}$, which gives (10.12). Moreover, since there exists $\epsilon > 0$ such that

$$\frac{1}{\tan((1-h_1)x)} + \frac{1}{\tan((1+h_1)x)} \geqslant \frac{1}{2\tan((1+h_1)x)} \qquad (x \in (n_k\beta, n_k\beta + \epsilon))$$

we have that there exists $r > 0$ such that

$$y_{n_k+1} - x_k \geqslant y_{n_k+1} - n_k)\beta > y_{n_k+1}^+ - n_k\beta > \frac{r}{k},$$

and (10.13) holds too.

In order to prove (10.14) notice that there exists $\epsilon > 0$ such that

$$\frac{1}{2\tan((1+h_1)x)} \leqslant \frac{1}{\tan((1+h_1)x)} + \frac{1}{\tan((1-h_1)x)} = x \qquad (x \in ((n-1)\beta, (n-1)\beta + \epsilon)).$$

Hence, by using Lemma 10.3, we have that

$$y_n - (n-1)\beta > y_n^+ - (n-1)\beta > r,$$

and the proof of the lemma ends. $\square$

The following theorem gives important information concerning the spectrum of the operator A_1 from Theorem 10.1.

Theorem 10.5. *Let $h_1 \in (-1,1) \setminus \mathbb{Q}$. The sequence of eigenvalues $(\lambda_m)_{m\geqslant 1}$ of the operator A_1 is the union of two increasing subsequences of positive numbers, $(\lambda_n^1)_{n\geqslant 1}$ and $(\lambda_k^2)_{k\geqslant 1}$, with the property that there exist $c_1, r > 0$ depending only on* $\min\{1-h_1, 1+h_1\}$*, for which we have*

$$|\lambda_n^1 - (n-1)^2\beta^2| \leqslant c_1 n \qquad (n \geqslant 1), \tag{10.15}$$

$$|\lambda_k^2 - k^2\alpha^2| \leqslant c_1 k \qquad (k \geqslant 1), \tag{10.16}$$

$$\inf_{m\geqslant 1} |\lambda_{m+1} - \lambda_m| > r. \tag{10.17}$$

Proof. We obtain (10.15)-(10.17) directly from Theorem 10.1 , Lemma 10.2, and Lemma 10.4. $\square$

We end this section with an estimate of the quantities $D(\lambda_n)$ defined by (10.5). In order to do this, we need to consider that h_1 belongs to the set of irrational algebraic numbers $\mathcal{S}$ introduced in (7.8).

Lemma 10.6. *Let $h_1 \in \mathcal{S}$. Then, for each $\varsigma > 0$, there exists a positive constant $M = M(h_1, \varsigma)$ such that the following estimate holds*

$$|D(\lambda_n)| \leqslant M n^{2+2\varsigma} \qquad (n \geqslant 1). \tag{10.18}$$

Proof. Let $\varsigma > 0$ be given. Firstly, since the set of algebraic numbers forms a field, we have that $h_1 \in \mathcal{S}$ if and only if $\frac{\alpha}{\beta} \in \mathcal{S}$. From Roth's Theorem (see [5, Theorem I, p. 104]) we deduce that there is only a finite number of pairs of integers $(p,q) \in \mathbb{Z} \times \mathbb{N}^*$ such that

$$\left|\frac{\alpha}{\beta} - \frac{p}{q}\right| < q^{-2-\varsigma}.$$

Consequently, there exists a constant $M > 0$, depending on $\frac{\alpha}{\beta}$ (and thus on h_1), such that

$$\left|\frac{\alpha}{\beta} - \frac{p}{q}\right| \geqslant \frac{M}{q^{2+\varsigma}} \qquad (q \in \mathbb{N}^*,\ p \in \mathbb{Z}). \tag{10.19}$$

Inequality (10.19) allows us to estimate from below the distance between the elements of the sequences $(\alpha k)_{k\geqslant 1}$ and $(\beta n)_{n\geqslant 1}$. We recall that, given any $k \geqslant 1$, there exists a unique $n_k \geqslant 1$ such that $\beta(n_k - 1) < \alpha k < \beta n_k$. If we denote

$$l_k = \min\left\{k\alpha - (n_k-1)\beta, n_k\beta - k\alpha\right\},$$

we deduce from (10.19) that we have

$$l_k = k\beta \min\left\{\frac{\alpha}{\beta} - \frac{n_k - 1}{k}, \frac{n_k}{k} - \frac{\alpha}{\beta}\right\} \geqslant \frac{M}{k^{1+\varsigma}} \qquad (k \geqslant 1). \tag{10.20}$$

To show (10.18), we estimate $D(\lambda_m^j)$ for each $m \geqslant 1$ and $j \in \{1,2\}$. Since $D(\lambda_m^j)$ blows up when the distance between $\sqrt{\lambda_m^j}$ and an entire multiple of $\frac{\pi}{1\pm a}$ tends to zero, let us analyze the quantities

$$d_n^1 = \begin{cases} \sqrt{\lambda_n^1} - (n-1)\beta & n \neq n_k \\ \min\left\{\sqrt{\lambda_{n_k}^1} - (n_k-1)\beta, k\alpha - \sqrt{\lambda_{n_k}^1}\right\} & n = n_k \end{cases} \qquad (n \geqslant 0),$$

$$d_k^2 = \min\left\{\sqrt{\lambda_k^2} - k\alpha, n_k\beta - \sqrt{\lambda_k^2}\right\} \qquad (k \geqslant 1).$$

We recall that, for each $n \geqslant 1$, we have $\beta(n-1) < \sqrt{\lambda_n^1} < \beta n$ and, for each $k \geqslant 1$, we have $\alpha k < \sqrt{\lambda_k^2} < \beta n_k$ and $\beta(n_k - 1) < \sqrt{\lambda_{n_k}^1} < \alpha k$. Thus, for each $m \geqslant 1$ and $j \in \{1,2\}$, d_m^j gives the distance between $\sqrt{\lambda_m^j}$ and the sequence $(\alpha k)_{k\geqslant 1} \cup (\beta n)_{n \geqslant 1}$.

Since Lemma 10.2 tells us that there exists $m_0 > 0$ with the property that

$$0 < d_m^i < \frac{\pi}{2} \qquad (m \geqslant m_0,\ i \in \{1,2\}), \tag{10.21}$$

we deduce immediately that

$$\left|D(\lambda_m^j)\right| \leqslant \tfrac{\pi^2}{(d_m^j)^2} \qquad (m \geqslant 1,\ j \in \{1,2\}). \tag{10.22}$$

We evaluate the quantities d_m^j by analyzing the following cases:

1. For $n \neq n_k$, by using (10.14), we deduce that
$$d_n^1 = \sqrt{\lambda_n^1} - (n-1)\beta \geqslant \frac{r}{n} \qquad (n \geqslant 1,\ n \neq n_k,\ k \geqslant 1). \tag{10.23}$$

2. To evaluate $d_{n_k}^1$ we note that $\max\left\{k\alpha - \sqrt{\lambda_{n_k}^1}, \sqrt{\lambda_{n_k}^1} - (n_k - 1)\beta\right\} \geqslant \frac{l_k}{2}$ and, by taking into account that $\sqrt{\lambda_{n_k}^1}$ is a root of (10.1), we deduce that there exists a constant $\tilde{d} > 0$ independent of k such that
$$d_{n_k}^1 \geqslant \frac{\tilde{d}}{k + \frac{1}{l_k}} \qquad (k \geqslant 1). \tag{10.24}$$
Indeed, from (10.7), (10.1), the definitions of $d_{n_k}^1$ and l_k, it follows that there exist positive constants $\tilde{d}_1$ and $\tilde{d}_2$ such that
$$\frac{\alpha}{2}k \geqslant \frac{\sqrt{\lambda_{n_k}^1}}{2} = \frac{1}{\tan\left((1-h_1)\sqrt{\lambda_{n_k}^1}\right)} + \frac{1}{\tan\left((1+h_1)\sqrt{\lambda_{n_k}^1}\right)} \geqslant \frac{\tilde{d}_1}{d_{n_k}^1} - \frac{\tilde{d}_2}{l_k},$$
from which we deduce immediately (10.24).

3. The same argument as above, allows us to deduce that there exists a constant $\tilde{d} > 0$, independent of k, such that
$$d_k^2 \geqslant \frac{\tilde{d}}{k + \frac{1}{l_k}} \qquad (k \geqslant 1). \tag{10.25}$$

Now, by taking into account (10.22)-(10.25), it follows that there exists a positive constant M, depending on h_1, such that

$$\begin{aligned} &\left|D(\lambda_n^1)\right| \leqslant \tfrac{M}{n^2} \qquad (n \geqslant 1,\ n \neq n_k,\ k \geqslant 1), \\ &\max\left\{\left|D(\lambda_{n_k}^1)\right|,\ \left|D(\lambda_k^2)\right|\right\} \leqslant M\left(\max\{k, \tfrac{1}{l_k}\}\right)^2 \qquad (k \geqslant 1). \end{aligned} \tag{10.26}$$

From (10.20) and (10.26) we deduce that (10.18) holds and the proof of the Lemma ends. □

Remark 10.7. *Note that the constants c_1 and r from Theorem 10.5 depend only on the distance $\min\{1-h_1, 1+h_1\}$ between the point h_1 and the extremities on the interval $[-1,1]$, whereas the constant M from Lemma 10.6 depends also on the diophantine approximation properties of h_1.*

11 Appendix B: Construction of a biorthoghonal family to a set of exponentials

Given $\alpha > \beta > 0$, let us consider two families of positive real numbers, $(\lambda_n^1)_{n\geqslant 1}$ and $(\lambda_n^2)_{n\geqslant 1}$, for which there exist some positive constants c_1 and r such that the following hypotheses hold

- (Λ_1) $|\lambda_n^1 - \beta^2 n^2| \leqslant c_1 n, \ |\lambda_n^2 - \alpha^2 n^2| \leqslant c_1 n \qquad (n \geqslant 1);$
- (Λ_2) $\sqrt{\lambda_{n+1}^1} - \sqrt{\lambda_n^1} \geqslant r, \ \sqrt{\lambda_{n+1}^2} - \sqrt{\lambda_n^2} \geqslant r \qquad (n \geqslant 1);$
- (Λ_3) $\dfrac{r}{k} \leqslant \inf_{n\geqslant 1} \left| \sqrt{\lambda_k^2} - \sqrt{\lambda_n^1} \right| \qquad (k \geqslant 1).$

The aim of this Appendix is to show that, for any $T > 0$, there exists a biorthogonal sequence to the family $(e^{\lambda_n^1 t})_{n\geqslant 1} \cup (e^{\lambda_n^2 t})_{n\geqslant 1}$ in $L^2\left[-\frac{T}{2}, \frac{T}{2}\right]$. Under hypothesis (Λ_1)-(Λ_2), it is known that there exists biorthogonal sequences to each of the families $(e^{\lambda_n^1 t})_{n\geqslant 1}$ and $(e^{\lambda_n^2 t})_{n\geqslant 1}$, separately (see, for instance, Tenenbaum and Tucsnak [26]). However, it is not completely obvious to show that the same is true for the union of these families. In order to do that, the separability condition (Λ_3) plays a fundamental role.

In this Appendix c denotes a positive constant which may change from one line to another and depends only of c_1 and r. Firstly, we present a very technical but important lemma which will be used later on.

Lemma 11.1. *Let $T > 0$ and $(\lambda_n^1)_{n\geqslant 1}$, $(\lambda_n^2)_{n\geqslant 1}$ be two families of eigenvalues which verify $(\Lambda_1)-(\Lambda_3)$. Then there exist entire functions $(G_n^1)_{n\geqslant 1}$, $(G_n^2)_{n\geqslant 1}$ and G_0^1 with the following properties*

1. *$(G_n^1)_{n\geqslant 1}$, $(G_n^2)_{n\geqslant 1}$ and G_0^1 are entire functions of exponential type less than $\frac{T}{2}$.*
2. *$G_n^1(i\lambda_m^1) = \delta_{mn}$, $G_n^1(0) = 0$ and $G_n^1(i\lambda_m^2) = 0$, $\forall m, n \geqslant 1$;*
3. *$G_n^2(i\lambda_m^1) = 0$, $G_n^2(0) = 0$ and $G_n^2(i\lambda_m^2) = \delta_{mn}$, $\forall m, n \geqslant 1$;*
4. *$G_0^1(i\lambda_n^1) = G_0^1(i\lambda_n^2) = 0$ and $G_0^1(0) = 1$, $\forall n \geqslant 1$;*
5. *$(G_n^1)_{n\geqslant 1}$, $(G_n^2)_{n\geqslant 1}$ and G_0^1 belong to $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ and there exist three positive constants c, ω and κ independent of n and T such that we have*

$$\|G_n^j\|_{L^1(\mathbb{R})} \leqslant \frac{c}{\lambda_n^j} e^{\omega\sqrt{\lambda_n^j} + \frac{\kappa}{T}} \qquad (n \geqslant 1, j \in \{1, 2\}), \tag{11.1}$$

$$\|G_0^1\|_{L^1(\mathbb{R})} \leqslant c e^{\frac{\kappa}{T}}. \tag{11.2}$$

The constants c, ω and κ are uniform for the class of sequences $(\lambda_n^1)_{n\geqslant 1}$ and $(\lambda_n^2)_{n\geqslant 1}$ verifying $(\Lambda_1) - (\Lambda_3)$.

Proof. For each $k \geqslant 1$ and $z \in \mathbb{C}$ we define

$$\phi_k^1(z - \lambda_k^1) = \prod_{n\neq k} \frac{\lambda_n^1 - z}{\lambda_n^1 - \lambda_k^1}, \ \phi_k^2(z - \lambda_k^2) = \prod_{n\neq k} \frac{\lambda_n^2 - z}{\lambda_n^2 - \lambda_k^2}, \ \phi_0^1(z) = \prod_{n\geqslant 1} \frac{\lambda_n^1 - z}{\lambda_n^1}, \ \phi_0^2(z) = \prod_{n\geqslant 1} \frac{\lambda_n^2 - z}{\lambda_n^2}.$$

For each $z \in \mathbb{C}$ we consider

$$G_k^1(z) = \frac{iz\phi_k^1(-iz - \lambda_k^1)\phi_k^2(-iz - \lambda_k^2)(-iz - \lambda_k^2)H_\zeta(2z)}{(\lambda_k^1)^2(\lambda_k^2 - \lambda_k^1)H_\zeta(2i\lambda_k^1)\phi_k^2(\lambda_k^1 - \lambda_k^2)} \qquad (k \geqslant 1),$$

$$G_k^2(z) = \frac{iz\phi_k^1(-iz-\lambda_k^1)\phi_k^2(-iz-\lambda_k^2)(-iz-\lambda_k^1)H_\zeta(2z)}{(\lambda_k^2)^2(\lambda_k^1-\lambda_k^2)H_\zeta(2i\lambda_k^2)\phi_k^1(\lambda_k^2-\lambda_k^1)} \qquad (k \geqslant 1),$$

$$G_0^1(z) = \left(\sum_{n\geqslant 1}\left(\frac{-i}{\lambda_n^1}+\frac{-i}{\lambda_n^2}\right)z+1\right)\phi_0^1(-iz)\phi_0^2(-iz)H_\zeta(2z),$$

where $H_\zeta(z) = \frac{1}{\|\sigma_\nu\|_{L^1}}\int_{-1}^1 \sigma_\nu(t)e^{-i\zeta tz}\,\mathrm{d}t$ and

$$\sigma_\nu(t) = \begin{cases} e^{-\frac{\nu}{1-t^2}}, & |t|<1, \\ 0, & |t|\geqslant 1. \end{cases} \tag{11.3}$$

Let $T>0$ and $\delta>0$. We chose the parameters ζ and ν from the definition of H_ζ as follows

$$\frac{T}{4} < \zeta < \frac{T}{2}, \quad \nu = \frac{(\pi+\delta)^2}{\zeta}.$$

Then we can use estimates (4.4), (4.14) and (4.15) from [26] and we obtain that there exist $B>0$ and $c>0$ such that

$$|\phi_k^j(-ix-\lambda_k^j)| \leqslant c(\lambda_k^j+|x|)^B e^{\pi\sqrt{\frac{|x|}{2}}} \qquad (x\in\mathbb{R}, k\geqslant 1, j\in\{1,2\}), \tag{11.4}$$

$$|H_\zeta(2i\lambda_k^j)| \geqslant \frac{1}{11\sqrt{\nu+1}}e^{\zeta|\lambda_k^j|/\sqrt{\nu+1}} \qquad (k\geqslant 1, j\in\{1,2\}) \tag{11.5}$$

$$|H_\zeta(2x)| \leqslant c\sqrt{\nu+1}e^{3\nu/4-(\pi+\delta/2)\sqrt{2|x|}} \qquad (x\in\mathbb{R}), \tag{11.6}$$

$$|H_\zeta(z)| \leqslant e^{\zeta|y|} \qquad (z=x+iy,\ x,y\in\mathbb{R}), \tag{11.7}$$

$$|\phi_k^j(z)| \leqslant c(1+|z|)^B e^{\pi\sqrt{|z|}} \qquad (z\in\mathbb{C}, k\geqslant 1, j\in\{1,2\}). \tag{11.8}$$

Since $\zeta < \frac{T}{2}$ by using (11.7) and (11.8) we have

$$|G_k^j(z)| \leqslant ce^{T|z|/2} \qquad (z\in\mathbb{C}, k\geqslant 1, j\in\{1,2\}).$$

Thus $(G_k^1)_{k\geqslant 1}$ and $(G_k^2)_{k\geqslant 1}$ are of exponential type less than $\frac{T}{2}$.

The function $G_0^1(z)$ have the same property. Indeed, this follows immediately by taking into account (11.7) and the estimates

$$|\phi_0^j(-iz)| \leqslant \prod_{k\geqslant 1}\left|1+\frac{|z|}{\lambda_k^j}\right| \leqslant \exp\left(\frac{\pi\sqrt{|z|}}{r}\right) \ (z\in\mathbb{C}, j\in\{1,2\}). \tag{11.9}$$

A straightforward computation reveals the fact that properties $2-4$ are fulfilled. Let us prove (11.1) for $j=2$, the case $j=1$ being similar. Firstly, we note that

$$|G_k^2(x)| \leqslant \frac{x\sqrt{x^2+(\lambda_k^1)^2}\,|\phi_k^1(-ix-\lambda_k^1)|\,|\phi_k^2(-ix-\lambda_k^2)|\,|H_\zeta(2x)|}{(\lambda_k^2)^2(\lambda_k^2-\lambda_k^1)\,|H_\zeta(2i\lambda_k^2)|\,|\phi_k^1(\lambda_k^2-\lambda_k^1)|} \qquad (x\in\mathbb{R}, k\geqslant 1). \tag{11.10}$$

In the following we will obtain lower estimates for the product $\phi_k^1(\lambda_k^2-\lambda_k^1)$, for any $k\geqslant 1$.

For each $k\geqslant 1$, let $n_k, \tilde{n}_k$ be two natural numbers which verify $\sqrt{\lambda_{n_k}^1} \leqslant \sqrt{\lambda_k^2} \leqslant \sqrt{\lambda_{n_k+1}^1}$, $2\lambda_{\tilde{n}_k}^1 - \lambda_k^2 - \lambda_k^1 \leqslant 0$ and $2\lambda_{\tilde{n}_k+1}^1 - \lambda_k^2 - \lambda_k^1 \geqslant 0$.

Note that, from $(\Lambda_1)-(\Lambda_3)$ it follows that there exist two constants $c, \tilde{c} > 1$ such that

$$ck \leqslant \tilde{n}_k \leqslant n_k \leqslant \tilde{c}k \qquad (k\geqslant 1). \tag{11.11}$$

Taking into account that

$$|\phi_k^1(\lambda_k^2-\lambda_k^1)| = \prod_{\substack{n=1\\ n\neq k}}^{\infty}\left|\frac{\lambda_n^1-\lambda_k^2}{\lambda_n^1-\lambda_k^1}\right| = \prod_{n=1}^{k-1}\left(1+\frac{\lambda_k^2-\lambda_k^1}{\lambda_k^1-\lambda_n^1}\right)\prod_{n=k+1}^{\tilde n_k}\left(1+\frac{\lambda_k^2+\lambda_k^1-2\lambda_n^1}{\lambda_n^1-\lambda_k^1}\right)$$

$$\underbrace{\prod_{n=\tilde n_k+1}^{n_k}\left(1-\frac{2\lambda_n^1-\lambda_k^1-\lambda_k^2}{\lambda_n^1-\lambda_k^1}\right)}_{Q_k^1}\underbrace{\prod_{n=n_k+1}^{\infty}\left(1-\frac{\lambda_k^2-\lambda_k^1}{\lambda_n^1-\lambda_k^1}\right)}_{Q_k^2},$$

we have that

$$|\phi_k^1(\lambda_k^2-\lambda_k^1)| \geqslant Q_k^1 Q_k^2. \tag{11.12}$$

By using $(\Lambda_1)-(\Lambda_3)$ we have that there exists $c>0$ such that

$$Q_k^1 = \prod_{n=\tilde n_k+1}^{n_k}\left(\frac{\lambda_k^2-\lambda_n^1}{\lambda_n^1-\lambda_k^1}\right) \geqslant \frac{\lambda_k^2-\lambda_{n_k}^1}{\lambda_{n_k}^1-\lambda_k^1}\prod_{n=\tilde n_k+1}^{n_k-1}\left(\frac{\lambda_{n_k}^1-\lambda_n^1}{\lambda_n^1}\right) \geqslant \frac{\lambda_k^2-\lambda_{n_k}^1}{\lambda_{n_k}^1-\lambda_k^1}\prod_{n=\tilde n_k+1}^{n_k-1}\left(\frac{r(n_k-n)}{cn}\right)$$

$$\geqslant \frac{\lambda_k^2-\lambda_{n_k}^1}{\lambda_{n_k}^1-\lambda_k^1}\left(\frac{r}{c}\right)^{n_k-\tilde n_k-1}\frac{(n_k-\tilde n_k-1)!}{n_k^{n_k-\tilde n_k-1}},$$

Hence we have that

$$Q_k^1 \geqslant \frac{\lambda_k^2-\lambda_{n_k}^1}{\lambda_{n_k}^1-\lambda_k^1}\left(\frac{r}{c}\right)^{n_k-\tilde n_k-1}\frac{(n_k-\tilde n_k-1)!}{n_k^{n_k-\tilde n_k-1}} \geqslant \exp(-ck). \tag{11.13}$$

where for the last inequality we have used (11.11) and Stirling's formula, $\lim_{n\to\infty}\frac{n!e^n}{n^n\sqrt{2\pi n}}=1$.

By using again $(\Lambda_1)-(\Lambda_3)$ we deduce that there exists $c>0$ such that

$$Q_k^2 = \prod_{n=n_k+1}^{\infty}\left(\frac{\lambda_n^1-\lambda_k^2}{\lambda_n^1-\lambda_k^1}\right) \geqslant \frac{\lambda_{n_k+1}^1-\lambda_k^2}{\lambda_{n_k+1}^1-\lambda_k^1}\prod_{n=n_k+2}^{\infty}\left(\frac{\lambda_n^1-\lambda_{n_k+1}^1}{\lambda_n^1}\right)$$

$$\geqslant \prod_{n=n_k+2}^{ck}\left(\frac{\lambda_n^1-\lambda_{n_k+1}^1}{\lambda_n^1}\right)\exp\left(\int_{ck}^{\infty}\ln\left(1-\frac{c^2k^2}{t^2}\right)\,\mathrm{d}t\right)$$

$$\geqslant \frac{\lambda_{n_k+1}^1-\lambda_k^2}{\lambda_{n_k+1}^1-\lambda_k^1}\prod_{n=n_k+2}^{ck}\left(\frac{r(n-n_k-1}{cn}\right)\exp(-ck)$$

$$\geqslant \exp(-ck)\left(\frac{r}{c}\right)^{ck-n_k-1}\frac{(ck-n_k-1)!}{(ck)^{ck-n_k-1}} \geqslant \exp(-ck).$$

Hence we have that

$$Q_k^2 \geqslant \exp(-ck)\left(\frac{r}{c}\right)^{ck-n_k-1}\frac{(ck-n_k-1)!}{(ck)^{ck-n_k-1}} \geqslant \exp(-ck). \tag{11.14}$$

By using (11.12)-(11.14) we have that there exists $\omega>0$ such that we have

$$|\phi_k^1(\lambda_k^2-\lambda_k^1)| \geqslant \exp(-\omega\sqrt{\lambda_k^2}) \qquad (k\geqslant 1). \tag{11.15}$$

In a similar way we can obtain that there exists $\omega>0$ such that we have

$$|\phi_k^2(\lambda_k^1-\lambda_k^2)| \geqslant \exp(-\omega\sqrt{\lambda_k^1}) \qquad (k\geqslant 1). \tag{11.16}$$

Since $|\lambda_k^1 - \lambda_k^2| > c$, by using estimates (11.4)-(11.6) and (11.15)-(11.16) in (11.10), we deduce that for any $k \geqslant 1$ and $j \in \{1,2\}$ we have

$$|G_k^j(x)| \leqslant c(\nu+1)\left(|x| + \lambda_k^j\right)^{2B+4} \exp\left(3\nu/4 - \delta\sqrt{|x|/2} - \zeta\lambda_k^j/\sqrt{\nu+1} + \omega\sqrt{\lambda_k^j}\right). \qquad (11.17)$$

Thus, from (11.17) we have that $G_k^j \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, for each $k \geqslant 1$ and $j \in \{1,2\}$.

Let $\gamma > 0$ be a constant sufficiently large, to be chosen latter on. We analyze the following cases

Case 1. $\lambda_k^j \leqslant \left(\frac{\gamma}{T}\right)^3$. We have that

$$\int_{\mathbb{R}} |G_k^j(x)|\,\mathrm{d}x \leqslant c(\nu+1)e^{\frac{3\nu}{4}+\omega\sqrt{\lambda_k^j}} \int_{\mathbb{R}} \left(|x| + \lambda_k^j\right)^{2B+4} \exp\left(-\delta\sqrt{|x|/2}\right)\,\mathrm{d}x \leqslant$$

$$\leqslant c(\nu+1)e^{\frac{3\nu}{4}+\omega\sqrt{\lambda_k^j}} \frac{1}{T^{2B+4}} \int_{\mathbb{R}} \left(T^3|x|+1\right)^{2B+4} \exp\left(-\delta\sqrt{|x|/2}\right)\,\mathrm{d}x \leqslant \frac{ce^{\frac{\kappa}{T}+\omega\sqrt{\lambda_k^j}}}{\lambda_k^j}.$$

Case 2. $\lambda_k^j > \left(\frac{\gamma}{T}\right)^3$. We have that

$$\int_{\mathbb{R}} |G_k^j(x)|\,\mathrm{d}x \leqslant c(\nu+1)e^{\frac{3\nu}{4}+\omega\sqrt{\lambda_k^j}} \int_{\mathbb{R}} \left(|x| + \lambda_k^j\right)^{2B+4} \exp\left(-\delta\sqrt{|x|/2} - cT^{\frac{3}{2}}\lambda_k^j\right)\,\mathrm{d}x \leqslant$$

$$\leqslant c(\nu+1)e^{\frac{3\nu}{4}+\omega\sqrt{\lambda_k^j}}(\lambda_k^j)^{2B+4} \int_{\mathbb{R}} \left(\frac{|x|}{\lambda_k^j}+1\right)^{2B+4} \exp\left(-\delta\sqrt{|x|/2} - c\gamma^{\frac{3}{2}}\sqrt{\lambda_k^j}\right)\,\mathrm{d}x \leqslant \frac{c}{\lambda_k^j}e^{\frac{\kappa}{T}+\omega\sqrt{\lambda_k^j}},$$

where the last inequality takes place for γ chosen sufficiently large such that the inequality $x^{2B+4} \leqslant e^{c\gamma^{3/2}\sqrt{x}}$ holds for any $x \geqslant 0$.

Hence, we have proved that (11.1) holds. In order to prove (11.2) we use (11.9) and (11.6) and we have that

$$|G_0^1(x)| \leqslant c\sqrt{\nu+1}x \exp\left(\left(\frac{2\pi}{r} - \sqrt{2}\pi - \frac{\delta}{\sqrt{2}}\right)\sqrt{|x|} + \frac{3\nu}{4}\right) \qquad (x \in \mathbb{R}, \qquad (11.18)$$

and for δ sufficiently large we deduce that $G_0^1 \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$. Finally, it follows that

$$\int_{\mathbb{R}} \left|G_0^1(x)\right|\,\mathrm{d}x \leqslant ce^{\frac{\kappa}{T}},$$

and the proof of lemma ends.

□

Based on the previous lemma, the following theorem gives a biorthogonal sequence to the union of families of exponential functions $(e^{\lambda_n^1 t})_{n\geqslant 1} \cup (e^{\lambda_n^2 t})_{n\geqslant 1}$.

Theorem 11.2. *Let $T > 0$, $(\lambda_n^1)_{n\geqslant 1}$ and $(\lambda_n^2)_{n\geqslant 1}$ be two sequences of positive numbers which verify properties $(\Lambda_1) - (\Lambda_3)$ and $\lambda_0 = 0$. Then there exist $(F_n^1)_{n\geqslant 1} \cup (F_n^2)_{n\geqslant 1} \cup \{F_0^1\} \subset \mathcal{C}^\infty\left[-\frac{T}{2}, \frac{T}{2}\right]$ which form a biorthogonal sequence to the family $(e^{\lambda_n^1 t})_{n\geqslant 1} \cup (e^{\lambda_n^2 t})_{n\geqslant 1} \cup \{e^{\lambda_0 t},\}$ in $L^2\left[-\frac{T}{2}, \frac{T}{2}\right]$ such that*

$$\|F_n^j\|_{\mathcal{C}\left[-\frac{T}{2},\frac{T}{2}\right]} \leqslant \frac{c}{\lambda_n^j}e^{w\sqrt{\lambda_n^j}+\frac{\kappa}{T}} \qquad (n \geqslant 1, \quad j \in \{1,2\}),$$

$$\|F_0^1\|_{\mathcal{C}\left[-\frac{T}{2},\frac{T}{2}\right]} \leqslant ce^{\frac{\kappa}{T}},$$

where the constants c, ω and κ are independent of n and T and uniform for the class of sequences $(\lambda_n^1)_{n\geqslant 1}$ and $(\lambda_n^2)_{n\geqslant 1}$ verifying $(\Lambda_1) - (\Lambda_3)$.

Proof. By using Lemma 11.1 we can apply Paley-Wiener's Theorem to deduce that there exists $(F_n^j)_{n\geqslant 0}$ from $L^2[-\frac{T}{2},\frac{T}{2}]$ such that

$$G_n^j(z) = \int_{-\frac{T}{2}}^{\frac{T}{2}} F_n^j(t)e^{-izt}\,\mathrm{d}x \qquad (n \geqslant 0, \quad j \in \{1,2\}). \tag{11.19}$$

From properties $2-5$ in Lemma 11.1 we deduce that $(F_n^1)_{n\geqslant 1}\cup(F_n^2)_{n\geqslant 1}\cup\{F_0^1\}$ is a biorthogonal sequence to the family $(e^{\lambda_n^1 t})_{n\geqslant 1}\cup(e^{\lambda_n^2 t})_{n\geqslant 1}\cup\{e^{\lambda_0 t}\}$. Moreover, from estimates (11.1)-(11.2) it follows that there exist three positive constants c, ω and κ independent of n and T such that

$$\|F_n^j\|_{L^\infty\left[-\frac{T}{2},\frac{T}{2}\right]} \leqslant \|G_n^j\|_{L^1(\mathbb{R})} \leqslant \frac{c}{\lambda_n^j} e^{w\sqrt{\lambda_n^j}+\frac{\kappa}{T}} \qquad (n \geqslant 1, \quad j \in \{1,2\}),$$

$$\|F_0^1\|_{L^\infty\left[-\frac{T}{2},\frac{T}{2}\right]} \leqslant \|G_0^1\|_{L^1(\mathbb{R})} \leqslant ke^{\frac{\kappa}{T}}.$$

Finally, the behavior of the entire functions G_n^j on the real axis, (11.17) and (11.18) imply that $F_n^j \in \mathcal{C}^\infty\left[-\frac{T}{2},\frac{T}{2}\right]$ for each $n \geqslant 1$ and $j \in \{1,2\}$. □

Theorem 11.2 allows us to deduce the existence of a biorthogonal sequence to the family of exponential functions $\left(e^{\lambda_n t}\right)_{n\geqslant 0}$, where the exponents $(\lambda_n)_{n\geqslant 1}$ are the eigenvalues of our operator A_1 from Appendix A.

Corollary 11.3. *Let $(\lambda_n)_{n\geqslant 1}$ be the sequence of eigenvalues of the operator A_1 given by Theorem 10.1 and put $\lambda_0 = 0$. Then there exists a biorthogonal sequence $(F_n)_{n\geqslant 0} \subset \mathcal{C}\left[-\frac{T}{2},\frac{T}{2}\right]$ to the family of exponential functions $(e^{\lambda_n t})_{n\geqslant 0}$ in $L^2\left[-\frac{T}{2},\frac{T}{2}\right]$ such that there exist three positive constants c, ω and κ independent of T with*

$$\begin{aligned} \|F_n\|_{\mathcal{C}\left[-\frac{T}{2},\frac{T}{2}\right]} &\leqslant \frac{c}{\lambda_n} e^{\omega\sqrt{\lambda_n}+\frac{\kappa}{T}} \qquad (n \geqslant 1), \\ \|F_0\|_{\mathcal{C}\left[-\frac{T}{2},\frac{T}{2}\right]} &\leqslant ce^{\frac{\kappa}{T}}. \end{aligned} \tag{11.20}$$

Proof. Note that, according to Theorem 10.5, the sequence of eigenvalues $(\lambda_n)_{n\geqslant 1}$ is the union of two subsequences $(\lambda_n^1)_{n\geqslant 1}$ and $(\lambda_n^2)_{n\geqslant 1}$, verifying $(\Lambda_1)-(\Lambda_3)$ with $\alpha = \max\left\{\frac{\pi}{1-h_1},\frac{\pi}{1+h_1}\right\}$, $\beta = \min\left\{\frac{\pi}{1-h_1},\frac{\pi}{1+h_1}\right\}$ and the constants c_1 and r given by (10.15)-(10.17). Therefore, there exist the families $\left(F_n^1\right)_{n\geqslant 1}$, $\left(F_n^2\right)_{n\geqslant 1}$ and $\{F_0^1\}$ satisfying the conclusions of Theorem 11.2. The family $(F_n)_{n\geqslant 0}$ is obtained, after some rearrangement and renotation, from the union $\left(F_n^1\right)_{n\geqslant 1}\cup\left(F_n^2\right)_{n\geqslant 1}\cup\{F_0^1\}$. □

Remark 11.4. *The construction and evaluation of the biorthogonal sequence $(F_n)_{n\geqslant 0}$, depend only on the properties $(\Lambda_1)-(\Lambda_3)$ of the exponents introduced at the beginning of this Appendix. Since the sequence $(\lambda_n)_n$ of eigenvalues of A_1 verifies the properties $(\Lambda_1)-(\Lambda_3)$, with constants c_1 and r given by* (10.15)-(10.17) *from Theorem 10.5, it follows that the constants c, ω and κ from* (11.20) *depend only on c_1 and r. Consequently, from Remark 10.7 we deduce that the constants c, ω and κ from* (11.20) *depend only on* $\min\{1-h_1, 1+h_1\}$.

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